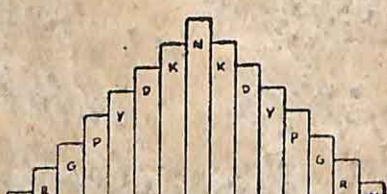
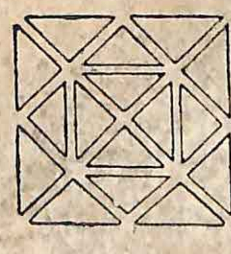
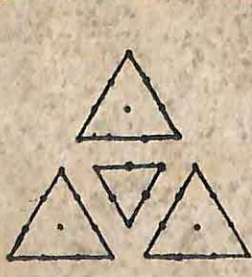
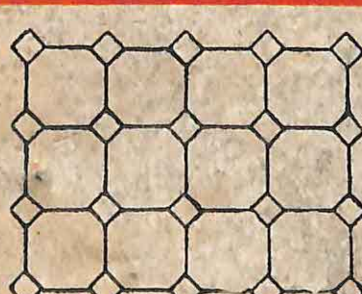
J. N. KAPUR

SUGGESTED EXPERIMENTS IN School Mathematics









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SUGGESTED EXPERIMENTS
IN
SCHOOL MATHEMATICS
[VOL. I]

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SUGGESTED EXPERIMENTS IN SCHOOL MATHEMATICS

VOLUME I

(*Experiments : 1—100*)

By

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The First Edition was
Dedicated to
The Elementary School Mathematics Teachers
of India

★

The Second Edition is
Dedicated to
My Mother Smt. Sukho Devi Kapur
who taught me my
First Lesson in Mathematics

PREFACE TO THE SECOND EDITION

The present edition differs from the first in the following respects :

(*i*) The experiments have now been grouped according to topics. The attempt at grouping revealed certain gaps and accordingly about ten new experiments have been added to fill these gaps.

(*ii*) Certain corrections and improvements have been carried out in the other experiments.

(*iii*) This volume is now the first of a book of three volumes of 500 suggested experiments in school mathematics. The adjective 'elementary' has therefore been removed.

(*iv*) The appendix on '36 basic principles underlying the experiments' has been considerably elaborated into a chapter on 'one hundred principles of mathematics teaching'.

(*v*) A set of specially-prepared one hundred exercises has been included. These exercises are of three kinds (*a*) exercises to test the understanding of the experiments given in the book, (*b*) exercises to suggest new experiments to the readers, (*c*) exercises aimed at making the readers 'think' in advance about some experiments included in the next two volumes.

(*vi*) A chapter explaining the basic philosophy of new mathematics has been included.

It is hoped that these changes will make the present edition considerably more useful than its predecessor.

During the last two years since the publication of the first edition, training programmes based on this book have been conducted for the following groups :

(*i*) 40 teachers of four Kanpur English-medium schools.

(*ii*) 15 teachers of Delhi, Rajasthan, U.P., Punjab and West Bengal deputed by the Council for the Indian School Certificate Examination.

(*iii*) 50 teachers of public schools of Mussorie.

(*iv*) 90 teachers of Kanpur Corporation schools.

(*v*) 40 parents of an English-medium school of Kanpur.

Some typical comments of the teachers who attended these courses have been the following : "Mathematics has become alive to us for the first time", "We envy our children who will be learning this exciting new mathematics", "It has been a thrilling experience for us".

Other activities in this connection which may be of interest to the readers have been the following :

(*i*) An exhibition of one hundred charts based on this book prepared by teachers of Kanpur schools.

(*ii*) Preparation of a set of charts and models by Kanpur Mathematics Study Group.

(*iii*) Preparation of teachers handbooks for teachers of classes I and II by the Kanpur Study Group and the testing of these books at the National Workshop on School Mathematics held at I.I.T., Kanpur and attended by representatives of eleven States.

(*iv*) Publication of Hindi edition of this book with translation of the author's article 'Revolution in Elementary School Mathematics' and with 50 pages of exercises (all different from those given in the present book).

(*v*) Publication of vol. II of this book with 200 more experiments and 100 more exercises.

(*vi*) Preparation of vol. III of this book.

(*vii*) Publication of my books 'Insight into Mathematics, Book I' (a text book for Class I) and its 'Teachers Guide' by the NCERT.

(*viii*) Publication of the 'Proceedings of the National Conference on School Mathematics by the newly formed 'Mathematical Association of India'.

(*ix*) Publication of my book 'New Mathematics for Parents'.

(*x*) Publication of my book 'Modern Mathematics for Teachers'.

It has been suggested, at the highest level, that school teachers may be encouraged to teach courses on new mathematics to

their students, on an optional basis and an all-India examination may be conducted to test the students and incentives in the form of prizes, book grants, certificates, scholarships may be given to the students, teachers and schools. It is believed that the three volumes of the present book will provide excellent texts for such a course.

I am grateful to Dr. R. R. Dikshit for his help in conducting the training programmes, to Shri J. P. Gupta for seeing the book through the press, to my publishers for the pains they have taken, to my colleagues of the Kanpur Study Group (Dr. S. K. Gupta, Dr. B. L. Bhatia and Shri P. K. Srinivasan) for their enthusiastic support, to members of the Study Groups at Delhi, Baroda, Bangalore, Jaipur, and Jadavpur for their cooperation, to principals of schools whose teachers have attended the training programmes, to Shri A. E. T. Barrow for deputing teachers to attend a training programme and to all the officers of the NCERT for their understanding cooperation.

I would be grateful to those who propagate the ideas in this book and who send me their suggestions for improvement.

I.I.T., Kanpur — J. N. KAPUR

15th Oct., 1969

PREFACE TO THE FIRST EDITION

The need for a revolution in elementary school mathematics was pointed out in a recent article.* This revolution has become necessary not only for the future of mathematics in India, but also for the future of all sciences and technology, since the habits formed in learning elementrary school mathematics are important for the entire learning and creative intellectual pursuits of the children. This revolution cannot however be willed into existence. For bringing it about, hard work, clear thinking, controlled experimentation and careful planning are necessary.

There is no doubt that creativity in children has to be encouraged and children have to be encouraged to recognize patterns and structures in mathematics. Learning by drill has to be replaced by learning by understanding of principles. Some of modern mathematical concepts have to be taught to school children, but these are to be presented in a manner entirely different from the way in which these are usually taught to college students. It is this fact, coupled with the recent psychological discovery that children can be taught almost anything provided it is taught properly, that has necessitated a great deal of research and thinking about (*i*) What concepts should be taught to children ? (*ii*) What concepts can be taught to children ? (*iii*) In what manner these concepts should be presented to children ? (*iv*) How are these concepts to be integrated ? (*v*) What audio-visual aids can be pressed into service ?

There are many teachers all over the country who are prepared to experiment with new ideas, but they need new ideas and help. The present volume is a first step to provide such help. We have suggested here one hundred experiments which can be tried by school teachers. This will be followed by two more volumes giving more experiments. The feedback from these experiments can be used to improve these experiments and ultimately lead to the development of a sound curriculum of 'new mathematics' suited to our schools.

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*J. N. Kapur, Revolution in elementary school mathematics, Science Resource Letter, Vol. 1. This article is also included in J. N. Kanpur's : "Some Aspects of School Mathematics".

For the success of these experiments, it is necessary that

(*i*) Teachers willing and enthusiastic for experimentation be available.

(*ii*) Proper training for these teachers through summer institutes, evening courses etc. be provided.

(*iii*) Proper supervision of these experiments be arranged.

(*iv*) Suitable audio-visual aids be devised.

(*v*) Experimental text books be written.

(*vi*) Proper encouragement and incentives for the teachers be given.

(*vii*) Scientific evaluation of results be made.

The class in which any particular experiment can be carried out has not been deliberately indicated, but has been left to the discretion of the teacher. In fact this will depend more on the calibre of the students than on the class to which they belong and the individual teacher knows best the calibre of his or her own students. At the present state of development of our school mathematics, some middle and secondary school teachers of mathematics may also find it useful to try the experiments with their students.

These experiments have been taught to students of English medium and Hindi medium schools and to groups of elementary school mathematics teachers. The reactions have so far been most encouraging. In fact in many cases the teachers started with some diffidence but as the training progressed, they became more and more enthusiastic until by the end of the programmes they became complete converts and showed a missionary zeal to implement these experiments.

It is hoped that the school managements, the municipal boards, State institutes of science education, State directorates of education, the national council of educational research and training, the national council of science education and other similar bodies interested in the development of school mathematics will conduct training programmes for these experiments. It is also hoped that the summer-school trained secondary school teachers will agree to help in this work and will even take the initiative in organising these training programmes. It is also desirable that the elementary school mathematics teachers decide to organise study groups to understand the full implications of these experiments. The training programme can be organised during summer or other vacations (a fortnight should be

more than enough), on Saturdays or Sundays or even in the evenings. The large scale adoption of these experiments will remove the present stagnation, give rise to meaningful and purposeful discussions and ultimately bring about a revolution in our school mathematics which is so vital for the development of our mathematics, science and technology.

At the end of this book, basic principles which have motivated the present experiments have been listed. The readers will find it worthwhile to refer to these again and again and to discover for themselves as to which experiments illustrate each principle and which principles are illustrated by each experiment.

This volume is being translated into Hindi and the translation will be available very shortly. A problem book is also being prepared to enable the students and teachers to have enough practice with the new ideas. The second volume containing more experiments is also under preparation. It is also hoped that it will soon be possible to prepare a set of standard charts and models to help the teachers in teaching these ideas.

I would welcome reactions from mathematics study groups, teachers and educational administrators.

I.I.T., Kanpur
25th July, 1967

J. N. KAPUR

CONTENTS

1

Introduction

1. The Need for Experiments in New Mathematics

School mathematics education is in ferment today. For the last twelve years or so, a revolution (picturesquely described as the 'New Mathematics Revolution') has been sweeping the world. This revolution has influenced hundreds of university mathematicians, thousands of college teachers, tens of thousands of school teachers and millions of children and parents all over the world. The revolution is in full swing today and from all accounts is going to gain momentum and continue indefinitely. The revolution is international in character and is the mathematicians' response to the fast changing world of science and technology and since science and technology are continuing to grow fast, school mathematics will have to continuously adjust itself in this fast changing world. The sooner this fact is realised by educationists, the better it will be for school mathematics.

However, the phrase 'New Mathematics' sometimes leads to a misunderstanding *viz.* that possibly either the earlier school mathematics was wrong or the mathematicians have found some new concepts which they are anxious to teach now to young children. The facts are as follows :

(*i*) There was nothing seriously wrong in the mathematics that was being taught in schools, though it is no use denying that there was something wrong with it.

(*ii*) The 'New Mathematics' is not new to mathematics though it is, to some extent, new for schools.

(*iii*) Even when the topics are not new, the method of presenting the topics is new.

(*iv*) Inspite of this, we continue to use the phrase 'New Mathematics' since it motivates the teachers and parents to learn, it gives the children an excitement of participation in a new programme and it prepares governments for investing more funds in mathematics education.

The main guidelines for this revolution are the following :

(*i*) Greater emphasis on mathematical structures ;

(*ii*) Greater emphasis on modern language of mathematics and modern concepts ;

(*iii*) Greater emphasis on modern applications of mathematics ;

(*iv*) Greater exploitation of the discoveries of the last two decades in psychology of learning ;

(*v*) Greater exploitation of teaching aids including computers ;

(*vi*) Integration of enrichment materials with the course to make mathematics learning an exciting experience for children ; and

(*vii*) A deliberate effort to bring the basic ideas of modern mathematics to the school class room.

The New Mathematics Revolution has involved a great deal of experimentation with new ideas. Some clarity has been obtained, yet the need for large scale experimentation continues. The need for experimentation under different cultural and educational conditions is equally important.

The present book is a contribution to this experimentation on a large scale. We suggest here 500 experiments which we believe will give the children the right perspective of mathematics, will make them active partners in the learning process and will make learning of mathematics as intellectually challenging, exciting and thrilling experience as it should be.

Ambitious teacher can carry out the experiments even without significant changes in the curricula. What is needed, of course, is an encouragement from the authorities. However, before a teacher tries out some of the ideas in the class room, he should understand them with either by careful self-study or through in-service training programmes. Where curricula have been changed, these experiments should give a deeper insight into the letter and spirit of the changes.

2. The Case for a Revolution in School Mathematics

(*i*) Mathematics is growing at an exponential rate and doubles itself in a period of about ten years so that we have today one hundred times as much mathematics as we had in the beginning of this century. This explosion in knowledge can be a threat to mathematics education unless we take steps in time to bring about revolutionary changes in curricula at all levels including the school level.

(*ii*) Most of the research in mathematics during this century has been motivated by the desire for clarity and deeper understanding of the nature of mathematics. This research has revealed many simplifying and unifying ideas and it is

but proper that we take these ideas into account specially when these ideas are as relevant to school mathematics as to college and university mathematics. This research has also revealed many general mathematical results and it is unfair to teach special results when more general results can be taught with the same effort.

(*iii*) During the last fifty years, mathematics has found new applications in industry, social, management and biological sciences and our curricula have to take these applications into account. We cannot behave as if mathematics and its applications have not grown beyond the nineteenth century level ; unfortunately most of our present day school mathematics is based on this assumption.

(*iv*) A number of new revolutionary discoveries in psychology of learning have been made during the last two or three decades *e.g.* we know how the children are capable of learning much more than we had thought them capable of, that some of the so-called advanced ideas of mathematics can be taught effectively to children, if these are presented properly, that children love to create and so on. In addition specific researches have been carried out in learning of mathematics by Piaget, Dienes, and others and we should make use of these results.

(*v*) We have today many teaching aids like charts, models, films, radio, television, slide rules, calculating machines and above all computers and we have to find the optimum methods for utilizing these teaching aids.

(*vi*) We have been completely static too long in a dynamic world. Our school syllabi have changed negligibly in spirit and content, during the last hundred years or so. It is not uncommon to find text books written forty or fifty years ago still being used in many schools. We have accordingly to accelerate our rate of progress at a revolutionary pace.

(*vii*) The tremendous explosion in mathematics first influenced M.Sc. syllabi, then B.Sc. syllabi, then higher secondary syllabi and now it is influencing school syllabi even at the primary level. This process was natural and inevitable and a stage has now come where wholesale, rather than piecemeal, changes have become necessary.

(*viii*) Revolution in school mathematics started in the West about twelve years ago. It reached our country about five years ago. The New Mathematics Revolution is a world-wide phenomena and it is going to have an important place in the history of science and mathematics.

(*ix*) School syllabi were framed earlier to meet the needs of the world of nineteenth century and, to some extent,

the needs of science and technology of that period. Since this world has tremendously changed due to space research, atomic research, computer revolution, cybernetic revolution, etc., mathematics syllabi have to change accordingly.

(*x*) Mathematics has too long been taught by rule-of-thumb methods. The modern world is changing so fast that rules of thumb have no place. The children who join schools now will come out as full citizens after fifteen or twenty years. Mathematics education should prepare them not for the world of 1850 or of 1890 or of 1920 or of 1960 or of 1970, but for the world of 1980 and 1990.

(*xi*) The computer revolution is making big changes in our life and the computer-generated mathematics has to be taken into account in school mathematics. Due to computers, a much larger proportion of the human race needs mathematical training today than that which required it even twenty years ago. Earlier we had to take into account mainly needs of physicists and engineers, today we have to take into account needs of medical and biological and social scientists and even of managers of industry. This also requires rethinking on a large scale.

(*xii*) There are today sharp discontinuities in the habits of thinking required by present day school mathematics, college mathematics and research mathematics. These discontinuities present difficult problems of adjustment to students. The need for N.M.R. arises due to the need for bringing about a uniformity of habit of thinking at all levels.

3. Some Notes on 'New Mathematics' and 'The New Mathematics Revolution'

(*i*) The NMR has not yet taken place, pressures for it are building up and one day it may start and when it starts, it is going to continue indefinitely. In other words, only the first phase of the NMR has started, the more serious second stage has yet to start.

(*ii*) The NMR so far has been inadequate to meet the needs of the times. The advance has been on the fringes. Discussions have been superficial and thinking on problems has been shallow. Serious revolutionary discoveries have yet to start.

(*iii*) One component of NMR is computer-assisted instruction (CAI) and teachers are just realising its importance.

(*iv*) For bringing about NMR, mass media of communication have to be used to the fullest extent.

(*v*) In NM, we have to project our plans twenty years in advance.

(*vi*) The NMR has to take into account, but has not to be inhibited by, what we are teaching today. Its ambitions need not be limited by what is possible today. It has to find out what is possible under ideal conditions *i.e.* with highly motivated students, highly trained and dedicated teachers, first rate text books and teaching aids and good physical conditions of learning etc. After finding what is realisable, we try to approximate to these conditions as much as possible. Under ideal conditions, it may be possible to teach present M.Sc. mathematics in schools and do it better.

(*vii*) In NMR, we believe in enabling children to connect many aspects of the world round them, to introduce them gradually to the processes of abstract thinking and foster in them a critical, logical but also creative turn of mind ; its main emphasis is on the learning process and on genuine understanding by children.

(*viii*) The NMR faces the enormously difficult problem of finding the best sequence of presenting mathematical ideas out of the millions of feasible sequences.

(*ix*) The final success of the NMR will be judged by the posterity, but the immediate success can be seen in the enthusiastic reaction of students and teachers.

(*x*) In future quite a good deal of elementary teaching will be done by machines and there will be some schools which will teach in schools what we teach in post-graduate classes today and will do a better job of it.

(*xi*) The school mathematics teacher of the future will have to be much better equipped and will have to be much more mathematically alive than his counterparts today.

(*xii*) The NMR sometimes results in loss of confidence by the school teacher. The society must do everything to give him help and assurance.

(*xiii*) Problem solving is as important in new mathematics as in the classical mathematics.

(*xiv*) Industry has a duty, in its own interest, to support and finance NMR.

(*xv*) NMR intends to find out the maximum possible acceleration for mathematics for a child and also to find out the maximum acceleration which should be given. The two are not the same.

(*xvi*) This is the first time in history when an overall view of curricular reform on a large scale in school mathematics has been considered and when the creative research mathematicians and the primary school teachers are cooperating in a common endeavour.

(*xvii*) The NMR sets great store by releasing the creative forces of the children.

(*xviii*) The NMR encourages research in mathematics education, it encourages experimentation and flexibility, it believes that there is a tremendous scope for improvement and every teacher should be free to do research in the laboratory of his class room.

(*xix*) The NMR is based on the premises that good mathematics can be learnt and enjoyed by a vast majority of children and not only by a small minority of talented children. If this was not true earlier, the fault lay not with the children, but with the educational system as a whole.

(*xx*) The NMR does not imply that teaching sets, modular arithmetic, number bases etc. will bring about a revolution. Much deeper changes are necessary.

(*xxi*) The NMR believes in the physical, intellectual and emotional involvement of children in the process of learning of mathematics.

(*xxii*) The NMR believes, that children can and should be made to appreciate the intrinsic nature of mathematics and the full import of its cultural significance.

(*xxiii*) In NM 'memorization of rules' and 'use of recipes' etc. are discouraged.

(*xxiv*) In NM effort is made to give the child the joy of pure knowledge and thus to persuade him to make the mental effort necessary for learning mathematics.

(*xxv*) The basic concepts that are emphasized in new mathematics are : structure, rigour, clearly defined vocabulary, correct proofs of theorem, generalisation, teachability and optional choice of topics.

(*xxvi*) In NM there is a feeling that while, text books are useful, even these tend to petrify syllabi and thus inhibit experimentation.

(*xxvii*) The NMR has brought home to every mathematician that school mathematics is also his responsibility.

(*xxviii*) The NMR has meant more a change in way of looking at topics already in the school than a change in the curriculum.

(*xxix*) The NMR tries to bring mathematics and reality closer together.

(*xxx*) In classical mathematics, there was a great emphasis on 'neat' problems which had 'nice' answers. In new mathematics, approximate or numerical mathematics of the 'ugly' variety has also a place.

(*xxxi*) In NM it is not enough to just write a good syllabus. There must be teachers who can inspire and who can create.

(*xxxii*) NM does not believe in 'symbol-showing' but in insight into what the symbols stand for.

(*xxxiii*) NM believes in using children's rich environment to give experiences for motivating the learning of mathematics by children.

(*xxxiv*) NM tries to demonstrate to the children how paper, pencil and thought alone suffice to give us undoubted power over the real world.

(*xxxv*) The whole subject of learning and teaching mathematics today is so exciting and so much full of possibilities that we are capable of making a real great advance.

(*xxxvi*) NM encourages schematic learning rather than rote learning.

(*xxxvii*) NM allows children to manipulate situations, grasp the regularities inherent therein and finally derive conclusions.

(*xxxviii*) NM tries to understand and remove the causes of 'number anxiety' and 'mathematical phobia' in children.

(*xxxix*) Abstract symbols and their abstract manipulation may be all right for the mathematicians. For the children these present many complex problems. NM makes an attempt to solve these problems.

(*xl*) In NM the first aim is to make the child appreciate the subject, then to give him manipulative skills and finally to coordinate these.

(*xli*) Some features of NM are : children must do the discovering themselves, the teacher must find their interest, must encourage discussion and practical work must precede computational work.

(*xlii*) In NM, it is believed that if children enjoy mathematics, they will find time to do it and they cannot help doing it.

(*xliii*) In NM a teacher asks a question, the pupils try to understand it, try to think about a possible answer and put it in words. Thinking and vision come first, words come afterwards. In earlier mathematics, it was all words and words.

(*xliv*) NM teachers realise mathematics is interesting, enjoyable and full of vitality and they want to pass on this enjoyment and enthusiasm to their children.

(*xlv*) The goal of NM is not only to increase the amount of knowledge of the child. It is to create opportunities for the child to invent and discover.

(*xlvi*) NM is easier for children than for teachers, for the former have nothing to unlearn.

(*xlvii*) In NM emphasis is more on mathematical literacy than on numeracy.

(*xlviii*) In NM the child must himself feel that he is growing and should not wait for the result of a test to find this out. He should have a feeling of accomplishment.

(*xlix*) In NM a teacher is a motivator and a source, rather than a mathematical indoctrinator.

(*l*) NM is child-centred rather than teacher-centred or text-book centred.

2

One Hundred Principles of Mathematics Teaching

In the remaining chapters of this volume and in all the chapters of the other volumes of this book, we shall be concerned mainly with the content of school mathematics. This chapter is, however, devoted to the pedagogic aspect. Good teaching is an art, but a teacher can always improve his teaching considerably by hard individual work, by deep interest in his students, by a dedication to teaching and by pursuing a long road of conscious individual experimentation and development. The following principles are meant to help the teacher in his noble task. The principles are based on the experiences of the author and of many other mathematicians, scientists, teachers and psychologists. The teacher interested in self-improvement will find it worthwhile to read these again and again and to interpret these in the light of his own experience. A good teacher should certainly be able to supplement these and improve these from his own experience. We now state the principles :

(*i*) Manipulative skills in mathematics cannot be obtained by doing a large number of drill-sums. These can be obtained only by understanding the structures of the systems under consideration. Drill very often impedes the learning process since it makes learning monotonous and it makes the subject look trivial. Thus the intellectual content of school arithmetic is very little and the rebellion of the children against arithmetic, which is very often ascribed to the difficulty of the subject, is really against the triviality of the subject.

(*ii*) Students have to be given appreciation of not only local structures in mathematics but also of global structure of mathematics. They have to study not only isolated facts, however interesting, but they have to study overall relations between facts. They must learn to get 'wholistic' or 'global' view of mathematics.

(iii) The 'spiral' approach should be followed. This implies that there should be repeated return to each topic in different classes, and each successive time we come to it, its treatment should become richer and deeper and its connections with other topics should be explored in greater details.

(iv) Mathematics should appear to the students as a work of genius of the human race and as one of the gradest intellectual edifices constructed by mankind. Mathematics is a great intellectual enterprise of the human race and the student should get a feeling for its intellectual achievements. They should however feel that it is a human achievement and they can also contribute to the development of mathematical culture.

(v) Mathematics is a dynamic enterprise. Students should know that some of the unsolved problems of the ancient times were solved in the nineteenth century, some of the unsolved problems of nineteenth century have been solved in the present century, there are a large number of unsolved problems in mathematics today and these are being solved by combined efforts of mathematicians from all over the world. Further they should know that in the process of solving of these problems, the mathematicians will come across more unsolved problems. This feeling of dynamism will remove some of the mystery which is attached to mathematics in the mind of the common man. This will also show that there can be no authoritarionism in the teaching of mathematics.

(vi) The best way to teach mathematics is to let the students recreate mathematics for themselves. Mathematics is learnt by doing it rather than by listening passively to it. Students should be helped to discover as much of mathematics as possible themselves. The teachers' role is to help the students in discovering mathematics.

(vii) Modern topics should be introduced, not just because they are modern, but because these can help to organise mathematics material for the students. In fact this is very often the case, since most important motivation for the creation of mathematics is always the need for better organisation of the existing mathematics.

(viii) Students should be taught rigorous mathematics but not to such an extent that they should begin to dislike sciences which are not so precise. The students' pleasure in logical precision of mathematics should be balanced by their urge to unravel the secrets of nature through mathematics.

(ix) Mathematics should be taught not as a finished product but as an evolving discipline. The students should be not

only able to deduce theorems from given axioms, but they should also be able to formulate axioms for a given mathematical system.

(*x*) The teaching of mathematics in a purely axiomatic fashion not only inhibits students from applying mathematics, it also gives a wrong picture of present day mathematics which is very often concerned with new situations both within and outside mathematics for which thinking required is quite different from what is required for axiomatic treatment.

(*xi*) The philosophy and nature of mathematical thinking should influence and be influenced by thinking around us. In mathematics, all operations are not commutative, but so is the case in ordinary life. In mathematics all order structures are not linear and partial orders are equally important, the same is true in everyday life. In mathematics we have to consider all possibilities, in life also neglect of any possibility leads to errors. Life deals with chance, mathematics develops a theory for the same. Mathematics teaching should aim at producing citizens who are rather precise in language, who are strong in their logical reasoning, who can handle abstract situations, who can generalise, who are not passive learners, who can detect fallacious reasoning, who can argue in depth, who have an appreciation for intellectual aesthetics and who have the willingness and ability to climb intellectual mountains.

(*xii*) In mathematics, intellectual effort can be a real time-saving device. Students should not be given algorithms, they should find algorithms.

(*xiii*) Mathematics should never be taught improperly since improperly taught mathematics may confuse the student, inhibit good mathematical reasoning and may even lead to dislike of the subject.

(*xiv*) Mathematics can be taught with different degrees of rigour. In earlier classes intuition should play a dominant role and as the student progresses, the dose of rigour should be increased gradually. In the spiral approach, same topic is taught with different degrees of rigour at different stages of the spiral. The student should feel conscious at every stage how his present treatment is more rigorous than what he had got earlier.

(*xv*) Even in teaching a topic in one class, degree of rigour can be gradually increased. The topic can be introduced in a purely intuitive way, its motivation can be based more on pre-mathematics than on mathematics and then gradually it should be made more precise, rigorous and abstract.

The student should be able to see how imprecisely stated concepts are gradually becoming more and more precise, more and more rigorous, more deep and more and more abstract. The teacher should enable the student to feel this growth of rigour, precision, depth and abstraction of a topic from class to class and also from day to day. The student should at the same time feel that the increase of rigour and abstraction means increase in power. The repetition of a topic in the spiral approach should not be boring, it should be really illuminating.

(*xvi*) When a concept is first introduced, it appears abstract to the student. A little familiarity with it makes it relatively concrete to him and prepares the ground for the introduction of a new idea which may appear again abstract to him and so on. It is similar to the experience we get when we move in a car. The distant objects appear dim, but as we approach them they become clearer and clearer and other objects at a distance appear dim now.

(*xvii*) The unity of mathematics should be emphasized throughout. Every opportunity should be taken to point out how different mathematical concepts are related with one another and how different concepts illuminate one another and how these relationships give great power to mathematics.

(*xviii*) In every piece of mathematics, we start with certain hypotheses and our aim is to prove some results. The student should be clear about the starting point as well as about the goal. He should be encouraged to think about possible paths of the goal. He should then be led towards the goal, but he should keep the goal always in view and see how the various steps he is taking lead him nearer and nearer to the goal. This sight of the goal will motivate the various steps, keep up his enthusiasm and prevent his being lost in the complexity of the various steps.

(*xix*) Many times he may be led to the goal by different paths and he may be helped to compare the elegance and depths of various paths.

(*xx*) Though the student may not be able to see himself the complete path from 'what is given' to 'what is to be proved', he may be encouraged to supply as many steps as possible.

(*xxi*) Though the share of rigour gradually increases in the consideration of earlier topics in mathematics, it must be realised that newer topics still have to be introduced more or less intuitively.

(*xxii*) In mathematics education, the student should build for himself a rich store house of intuitive experience with

mathematical situations. The axiomatic logical basis cannot be built up satisfactorily without this experience.

(*xxiii*) The teacher should employ multiple approaches in dealing with a subject. Each approach should supplement the other. The various approaches should preferably be at the same level of rigour. Different students will react to different approaches and perhaps everyone will find at least one approach suitable for himself. Different approaches may even lead to different generalisations and different further results. Some approaches may be simple but may not be capable of easy generalisation or may not be able to throw light on more general problems. Others may be more difficult but may have much wider applicability.

(*xxiv*) When an important theorem is proved, its motivation and its significance should be clearly brought out.

(*xxv*) Whenever both a constructive proof and a non-constructive proof are available, the constructive proof should always be preferred because it appeals more to the students.

(*xxvi*) One of the strongest words in the learning process is 'motivation'. Whatever mathematics is done should be as strongly motivated as possible.

(*xxvii*) The teacher should decide what the basic ideas in mathematics are which he wants to highlight, which are fundamental to the development of the subject and which he should emphasize again and again. He should build his teaching round these ideas.

(*xxviii*) What precision is to concepts, accuracy is to skills. For developing accuracy in the use of algorithms, checking is very essential. The student should be encouraged to develop his own checking rules by using the structure of the problem. The development of such rules will also lead to mathematical development on the part of the students.

(*xxix*) The discovery approach in which a student is asked to explore a mathematical situation on his own, develops creativity, independent thinking and selfconfidence in the student. His innate interest in his feeling that his discovery is his own leads to a great self-assurance, he is put into a competitive atmosphere with his class fellows and a greater concentration is obtained from him. His whole personality becomes completely absorbed in the learning process and he is not going to forget what he has learnt through this approach.

(*xxx*) The discovery approach is slow, but the aided discovery approach is relatively faster. The student's correct discovery must be appreciated and he should regard it as his moment of triumph. His half-correct ideas should be used to lead him to correct ideas and mistakes in his wrong ideas

should be sympathetically corrected. In no case the teachers should reject his correct solution simply because this solution does not agree with the teachers.

(*xxxi*) Class room discussion method is very useful. The student is not prepared to reject a teacher's suggestion, but he is prepared to examine very critically a suggestion made by a class-fellow of nearly the same intellectual calibre as his own. The ideas that arise in discussion are also nearer to his own level of thinking and thus are more easily acceptable to him.

(*xxxii*) Tricks and puzzles are often annoying to the students. To some students, mathematics even appears as a bundle of tricks. If however the theoretical bases of tricks and solutions of puzzles are examined carefully, it would lead to an increased understanding of mathematics and may throw welcome light on some otherwise obscure results.

(*xxxiii*) The role of exercises should be clearly understood. In fact one approach suggests that after giving the basic definitions, most of the results should be given as carefully graded exercises. After the students have attempted these exercises, their solutions should be discussed in the classroom.

(*xxxiv*) The historical approach to the teaching of mathematics may be quite rewarding. This may be done by (*a*) giving some anecdotes about the mathematicians whose names may occur in certain contexts, thus giving a human touch to the teaching ; (*b*) discussing how various mathematical concepts arose as products of the social and historical conditions of times ; (*c*) giving formal courses in history of mathematics ; and (*d*) emphasizing mathematical creativity of individuals and groups.

(*xxxv*) Discussion of unsolved problems will show mathematics as an open-ended discipline. It will provide motivation for learning mathematics to some students at least. A list of such problems will always be useful to the teachers.

(*xxxvi*) Audio-visual aids should be fully exploited in the teaching programme. These include Dienes multibase blocks, Cuisainaire rods, flannel graphs, geoboards etc. at the elementary level and charts, models, film strips, films, radio and television lessons at the higher level.

(*xxxvii*) The mathematics laboratory may prove a useful idea in teaching. It may consist of (*a*) experiments to discover patterns in numbers and forms ; (*b*) use of audio-visual aids to learn mathematics ; (*c*) use of programmed learning materials ; (*d*) mathematical games ; (*e*) charts, models and film strips ; (*f*) physical experiments to verify laws; (*g*) making mathematical models for physical situations ; and (*h*) making of charts and models by the students themselves.

(*xxxviii*) Enrichment material should constantly be used. There is a marked difference in calibre between bright and average students. Bright students can be given enrichment materials to work with.

(*xxxix*) Applications should always be used to motivate a topic both when it is introduced and when it is generalised. Both internal and external applications can be strong motivations for learning mathematics. Applications also require a greater depth of understanding and also greater ingenuity on the part of the students. Applications should however be within the experience of the student or the experience should be capable of being acquired easily. The applications should not also be trivial.

(*xl*) Formulation of mathematical models for unfamiliar situations, deducing the consequences and testing the validity of these consequences should be part of the teaching programme.

(*xli*) Every citizen has to use quantitative decision making. All efforts should be made to strengthen his capacity to do so. The transfer of learning of mathematical habit of thinking to other fields should be positively encouraged.

(*xlii*) The concepts of inequalities and orders of approximation should be introduced at an early stage. Orders of magnitudes calculation to check validity of answers are useful. In many situations precise numerical answers are not wanted and are not warranted by the input and output structure of the model. A mathematical model is an approximation to reality and the precision of calculations on the model should match the precision of the model itself. Reasonable 'rounding off' procedures should be repeatedly emphasized.

(*xliii*) If a mathematical idea appears not to be understood by the students, the teacher must examine carefully whether he has himself understood it clearly and whether he has presented it clearly. The teacher has to make a conscious effort to understand all the implications of the mathematical idea and its relation with other mathematical ideas. He should present the concept from as many different points of view as possible.

(*xliv*) All efforts to prove the completely plausible and obvious results lead to the feeling that in mathematics there is much ado about nothing. All efforts to prove the utterly implausible lead to the losing of faith in mathematics. Such results are to be avoided. However when these have to be proved, special pains have to be taken to explain the need for such results.

(*xlv*) A drill material can be presented by any drill master, but the presentation of concepts of mathematics in a challenging, stimulating and exciting fashion requires a great deal of effort in presentation on the part of the teacher. In order to elicit all the insights and creative responses from the student, the teacher has to be prepared to make such an effort. It has to be recognized that the limits on the insights and creativity of the students are set by the limitations of teachers and teaching materials rather by the incapacity of the students.

(*xlvi*) The teachers should know much more than what they have to teach. They should know enough to be able to answer any unexpected questions from enquiring bright minds and they should feel happy with such questions rather than feel embarassed by these.

(*xlvii*) Carefully selected sequences of problems are important aids to teaching. These should not only illustrate what has been taught, these should also motivate further learning and these should make the students think and sometimes even think hard. 'Discovery exercises' *i.e.* exercises which lead to discovery of patterns in mathematics and 'discovery problems' which lead students to discover mathematics for themselves and which make the students feel that mathematics is a field par excellence for creative activity, should be specially created and used at least with brighter students.

(*xlviii*) Students should learn what a mathematical proof really is. They should be able to distinguish between a discussion which is a proof and one which has only the appearance of a proof. The meaning of such phrases as 'necessary and sufficient conditions', 'existence and uniqueness theorems', 'implies and is implied by', 'if and only if', 'proof by reductio ad absurdum', 'proof by cases', 'proof by induction' etc. should be quite clear to the students.

(*xlix*) Student projects are also a great help to the learning process. These are longer problems which require hours and days, which require even independent reading by students and which have definite goals.

(*l*) Teaching must take into account the impact of computers on mathematics education. Iterative processes, flow diagram, approximations etc. should be emphasized throughout. Teachers should familiarise themselves with computer programming, programmed instruction and computer assisted instruction.

(*li*) Every teacher owes a debt to his profession. His teaching will be considerably strengthened by his membership of professional societies, his reading of professional journals, his contact with scholars and other similar activities.

(*lii*) Successful teaching has to be creative. Every teacher should regard himself as a research scholar in mathematics education. Every teacher has to believe in continuing education for himself. Every teacher has to have a feeling for the psychology of learning of mathematical concepts. Every teacher has a rich laboratory in his classroom for experiments in mathematics education.

(*liii*) The teaching programme should be so designed as to reduce the amount to be later unlearnt to a minimum since this process is painful both to the student and the teacher. The student should be given intuitive and plausibility considerations only when it is necessary and then he must know the true nature of these considerations. It should not be necessary to tell him 'what you were taught earlier was wrong, now we are going to give you the correct thing'.

(*liv*) An attitude of intellectual honesty should be developed among the students. The students should know what is being assumed and what has been proved, where there is lack of precision or lack of rigour. They should be aware of the imperfections in the arguments. They should not be indifferent to rigour, nor they should be paralysed by it. Discovery and proof are both vital in mathematics. Intuition and logic both have their places. The teacher should not tell lies to the students, but he should not tell them the whole truth for teaching the whole truth at one time will mean a completely rigorous treatment which is really out of question because of pedagogic difficulties.

(*lv*) The teacher must aim at quality both in his methods and content and this must be done irrespective of the fact whether students have desire to become mathematicians or users of mathematics or just intelligent citizens. Ugly mathematics has no place for any category of persons.

(*lvi*) Teaching by using the authority of the teacher or teaching by waving of hands or proofs by intimidation have no place in good mathematics teaching. The teacher himse f should be intellectually honest with his students.

(*lvii*) Teaching of mathematics should be integrated with the other subjects, as far as possible, specially in schools.

(*lviii*) The activities of students mathematics clubs and the teachers' time spent in directing the activities of these clubs should be considered essential components of the teaching process.

(*lix*) In teaching mathematics, global axiomitisation should be used whenever possible. When this is not possible, local axiomitisation should be fully exploited.

(*lx*) The goal of teaching mathematics is to make the students think in the same way in which mature mathematicians and

necessary repetition and summary. The teacher should guard against repeating the text book in the class. He should rather devote his time to make the subject more meaningful and stimulating by connecting the material with the students' previous knowledge in the field. The teacher should underscore the most important ideas and should clear up possible ambiguities.

(*lxix*) The teacher should encourage the students to read books and articles of the enrichment type from the library. He must, of course, himself read these regularly so that he may be able to make a careful selection.

(*lxx*) The teacher should not be over anxious to help a student who is struggling with a problem. The student may have to muddle around on a problem before it becomes clear to him. The muddling process is the core of creative thinking. The teacher may give hints but should not deny the student the chance to think.

(*lxxi*) The teacher should carefully look at the expressions on the faces of his students and he should go on adjusting his teaching to the feedback he thus receives. If he finds that the students are not following, he should repeat the material in a slightly different language and he should go slow. He may even occasionally make a deliberate slip and if he finds no student catches it, he can be sure that the class is not following him and he should take immediate remedial action.

(*lxxii*) The teacher should encourage questions both inside and outside the class room and he should not appear annoyed even if the questions are not relevant. He should explain the point and at the same time give even more challenging questions to the students He may even ask students to propose questions different from those given in the text book.

(*lxxiii*) The mathematics teacher should remember that while he may have been deeply immersed in his subject before he goes to the class, his students may have been reading general science or social studies before coming to his class. He should start with a brief review and clarify those points which he thinks he could not make quite clear in his previous lecture. He should start with new material after he and the students have got attuned together for new mathematics.

(*lxxiv*) The teacher should deliberately aim to arouse interest, stimulate the yearning to learn and gain undivided attention from the students for the new material and ideas. To do these things, the teacher must himself feel and exhibit enthusiasm for the subject. The student must feel that

scientists think. The achievement of this goal is perfectly feasible and is worth working for.

(*lxi*) The teacher should give knowledge of mathematics to the student in such a way that (*a*) he may be able to apply it to new situations, (*b*) he may recall it easily, (*c*) he may be able to correlate and organise his ideas, (*d*) he may be able to think critically and creatively, and (*e*) he may be able to read, write and even think in the language of mathematics.

(*lxii*) The teaching of mathematics should inculcate in the student, habits of quantitative thought, even in daily life. It should become a second habit with him to think quantitatively, to measure, to estimate and to compare.

(*lxiii*) The teacher should not aim to 'cover' a course in mathematics in complete details. He should 'uncover' a part of it and enable the student to uncover the rest of it himself with the help of the text book. The student should be encouraged to use the text book quite often. He should not regard it as just a compendium of problems.

(*lxiv*) Before starting a new course, the teacher (*a*) should find out whether the students really know all the prerequisites for the course, by giving a well-designed quiz, (*b*) should plan and procure all the teaching aids the teacher may like to use, and (*c*) should see that the physical arrangements in the class room are quite satisfactory.

(*lxv*) The mathematics teacher must insist on a good blackboard on which he can write effortlessly. He should see that there is no glare on the board, that the writing on the board is visible from every part of the room, that he does not cover the writing on the board with his body, that he writes bold enough letters and symbols on the board, that he starts writing from the left hand corner of the board and plans in advance the use of the writing space on the black board.

(*lxvi*) Before every period, the teacher should prepare carefully and adequately what he has to teach in that period and should even rehearse it, he should plan the motivation aspect and even provocative questions which may make the students think carefully in the class room.

(*lxvii*) Immediately after a lecture, the teacher may write a note as to how he might have made the lecture more interesting and exciting and he may consult this note before giving the same lecture next time.

(*lxviii*) The teacher should not try to cover every detail in the class-room. He should make a considered selection of the most important or most stimulating material, allowing time for questions and discussions by the students as well as for

what he is learning is very important. He should have been given sufficient motivation for learning. The student should feel that learning mathematics is a highly exciting and useful process.

(*lxxv*) The student must be made to see how the new topic is important, how the new topic is connected with what he already knows, how it will be related to what he is going to learn in mathematics and science. To learn effectively, the student must first have the desire to learn. He will have this desire if he understands why the new material is important and if he feels that he is making steady progress and he is getting somewhere. Nothing succeeds like success, even in the learning process.

(*lxxvi*) The teacher must go from the known to the unknown, from the familiar to the unfamiliar, from concrete to abstract, and from easy concepts to more difficult concepts. He should consolidate his gains as he goes along. He should define his terms carefully and state his theorems precisely. He should explain the meaning of each term and each condition separately. Counter examples are very useful for this purpose.

(*lxxvii*) The teacher should plan his examinations carefully to match clearly-stated objectives of teaching, to get a positive response from the students and to ensure reliability of results.

(*lxxviii*) The teacher must realise that for effective mathematics teaching, there has to be capital investment in the form of good black boards, charts, films, video-tapes, overhead projectors, duplicating machines and even computers (for computer-assisted education and for evaluating in detail the responses of the students). The teacher must take every opportunity to impress on the authorities the need for these, but their non-availability should not prevent him from putting in his best.

(*lxxix*) The teacher must realise that mathematics can be taught effectively in some intellectually honest form to students at any stage of development. However, it will require creative thinking on his part to bring advanced ideas to the school class room.

(*lxxx*) The teaching and learning of mathematical structures, rather than simply the mastery of isolated facts and techniques, is at the centre of the problem of transfer of learning in mathematics. Good teaching that emphasizes the structure of mathematics is possibly even more valuable to the less able students than for the gifted students.

(*lxxxi*) The teacher should aim at the optimum intellectual development of each child and should not direct his efforts at good students only.

(*lxxxii*) Every mathematics teacher must learn modern mathematics, for teaching without this knowledge may not be exciting.

(*lxxxiii*) The teacher should motivate intuition, should stress the value of shrewd guess and at the same time should emphasize the need for proofs.

(*lxxxiv*) If a teacher has a twinkle in his eyes, he gets a twinkle in the eyes of his students. A good teacher is one who teaches the least but draws power from his students. A teacher should have a rich and enquiring mind. He should ask why and not only how, he should seek satisfaction in accomplishment, he should be fully responsible and must have integrity, courage and high personal standards of performance. He must be dedicated to his students and to teaching.

(*lxxxv*) The teacher should emphasize and illustrate how in mathematics education, we continue steadily the process of learning to draw finer and finer distinctions between similar things and of organising them into more and more elaborate structures.

(*lxxxvi*) The teacher has to prepare his students for an unknown future, he should therefore emphasize the fundamentals of mathematics. He should prepare his students to grow with mathematics, he should give them a feeling for the continuous growth of mathematics and he should develop their understanding of the relation of mathematics to other sciences.

(*lxxxvii*) The teaching of mathematics cannot remain healthy if it gets stagnant, if it gets into a set routine, if fresh ideas are not accepted, specially at a time when mathematics knowledge is advancing so fast.

(*lxxxviii*) The present day teacher must realise that he is living on an exciting period of the development of mathematics education. At no time in history did the teaching of mathematics present more problems, make more demands or offer the promise of a richer satisfaction.

(*lxxxix*) The teaching of mathematics needs to be better integrated, with contemporary applications in science, industry and technology. In fact these applications should often be the vehicles by which the subject is introduced.

(*xc*) The students should learn the language of mathematics and must be able to give clear explanations of fundamental concepts, statements and notations. He should know the basic algorithm. He should appreciate the power of abstraction and of the axiomatic method. He should be aware of the applicability of mathematics and of the

constructive interaction between mathematics and the sciences. He should be able to read mathematical literature at his level with understanding and enjoyment. He should grow steadily in mathematical maturity.

(*xci*) Special care should be taken of the mathematically gifted pupils. They should be encouraged to read advanced literature, to sit in advanced classes and should have opportunity to discuss with their teachers. Mathematically gifted students are an asset of the nation and they must be treated as such.

(*xcii*) Creativity is the heart and soul of mathematics. It must take place in elementary classes, middle classes and high school classes. Curiosity and creativity are key-stones in mathematics.

(*xciii*) Mathematics education should provide understanding of the interaction between mathematics and reality, should make students understand that mathematics is based on intuitive understanding and agreed conventions which are not eternally fixed and that some complex things are sometimes quite simple and coversely.

(*xciv*) Since mathematics is a growing and dynamic subject, it is best taught by teachers who are mathematically alive and active, who read new text books and journals, who attend conferences, who write expository articles, who are making deliberate efforts to learn new mathematics, to teach new mathematics and even to create new mathematics.

(*xcv*) The teacher should emphasize the boundless wealth of deductions from the interplay of general theorems, the apparent remoteness of the deductions from the axioms, the intellectual nature of the subject and the great power of mathematics.

(*xcvi*) The teacher of mathematics must not allow himself to lose touch with the living branches of his subject, he must find time or make time to keep his teaching fresh and vigorous.

(*xcvii*) The teaching of mathematics should be so organised that the student gets acquainted with newer concepts as correctly and as early as possible.

(*xcviii*) It is easiest to teach modern mathematics to the very young children, for they have enquiring minds and their confidence has not been shaken by bad teaching.

(*xcix*) The mathematics teaching should aim to enable the student to appreciate the intrinsic nature of mathematics and the full import of its cultural and scientific significance.

(*c*) A teacher should carefully avoid the temptation to think that what he learnt at an advanced stage cannot be taught to young children. The sequence of topics in the present curriculum has not been rationally thought out and there is a great scope for improvement here.

3

Addition and Subtraction Operations

[EXPERIMENTS 1—30]

Experiment 1 : Recognition of Symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, +, −, =

Before children learn addition and subtraction, let them learn to recognize the symbols they are going to use. Four alternative methods for this purpose are given below :

Method 1 : Get small biscuits made of the shapes of these symbols; ask children to recognize these. The child who recognizes the largest number of symbols is declared the winner and gets a biscuit. The child who recognizes all, wins a full set and is eliminated from the game. The game continues till every child recognizes every symbol.

Method 2 : Write these symbols in random order on the board and ask the children to recognize these.

Method 3 : Get these symbols written on tin plates or card boards or bottle tops and hang these by small nails on the wall or on the board and ask the children to recognize these. Alternatively call a child, ask him* to choose a particular symbol from a heap lying on the table and hang it on the board. If the child fails, ask another.

Method 4 : Put all the symbols in a box or in a bag, ask each child to take one symbol from the box, show it to the class and speak it aloud.

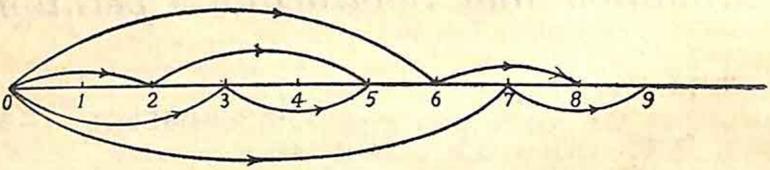
*Throughout this book 'him' will stand for 'him or her' and 'he' will stand for 'he or she'.

Experiment 2 : Some Methods for Teaching Addition

Some of the alternative methods available for this purpose are given below :

Method 1 : Adding with the help of the number line

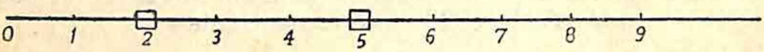
To find 2+3, the child draws an arrow for 0 to 2 and then he moves 3 more steps from 2 to reach 5. 5 is the sum of 2 and 3. Similarly let him find 3+2 and verify that 2+3=3+2.



The figure shows addition of 6 and 2 and of 7 and 2. The same process can be done without use of arrows. The child says '1', '2' and stops at 2. He places one finger of left hand on 2 and then moves 3 steps to the right, saying '1', '2', and '3' and then stops at 5. The number line may be painted on the board and the children may work with chalk. Coloured chalks may also be used *e.g.*, the first line may always be drawn in red and the second line in blue.

Method 2 : Adding with the help of magnetic counters

The number line may be drawn on an iron plate suspended on the board. Two or three magnetic counters of different colours may be made. To add 2 and 3, first one counter is placed at 2 and then the second counter is moved 3 steps to the right from 2.

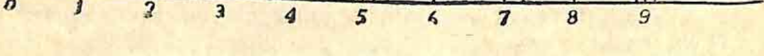


Now suppose the second counter is placed at 8. The first counter can be placed at 1 or 2 or 3 or 4 or 5 or 6 or 7 giving,

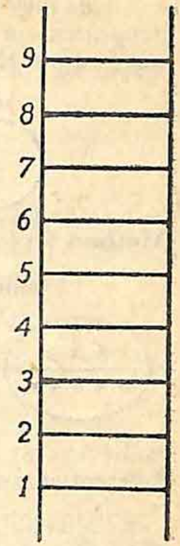
1+7=8, 2+6=8, 3+5=8, 4+4=8, 5+3=8, 6+2=8, 7+1=8.

Similarly place the second counter at 6 or 7 or 9 and ask children to place first counter in between and to read the addition sum. Three counters may also be used *e.g.*, the following figure shows

2+3+4=9.



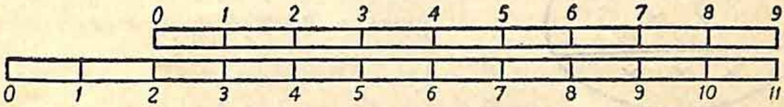
Method 3 : Adding with the help of a ladder



The ladder has nine steps. To find 3+4, the child counts '1', '2', '3' and goes up 3 steps, he again counts 1, 2, 3, 4 and stops at 7 to give 3+4=7.

The ladder may be made of iron and suspended on the wall or board. The counter may be made of a magnet in the shape of a child going up the ladder.

Method 4 : Adding with the help of two foot-rules

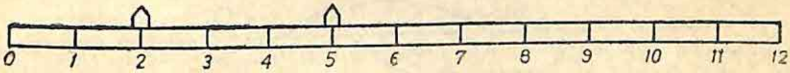


To add 2 and 3, we place the two foot-rules on the table as shown above and slide the upper one till its beginning coincides with the 2 mark on the lower one, we read 3 on the upper one and find the number on the lower one corresponding to 3 on the upper one. The number gives the sum of 2 and 3.

This gives the principle of the slide rule.

If there is difficulty with ordinary foot-rules, special foot-rules may be made. Centimetre scales can also be used.

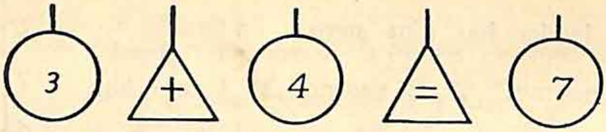
Alternatively a foot-rule with two sliders may be used.



Method 5 : Adding with the help of toys

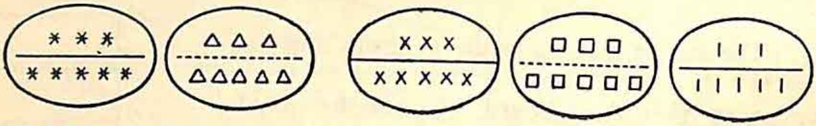
Toys can be placed on the table. Each child can be asked to make a group of 3 (or 4) horses and another group of 4 (or 5) horses and then asked to find the total number in the two groups and to say *e.g.*, that 3 and 4 make 7 (or 4 and 5 make 9).

He may pick up symbols for 3, +, 4, =, 7 from the table and hang them on the board and if magnetic symbols are available, he can first put these on the iron plate to read $3+4 = 7$.



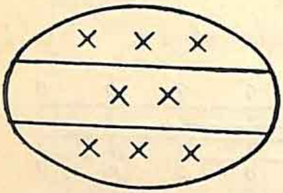
Method 6 : Adding with the help of symbols

The child may be taught to read each of the following :



as $3+5 = 8$. The two groups of symbols may be drawn by using different coloured pieces of chalk.

Three groups may also be used, *e.g.*;



gives $3+2+3 = 8$.

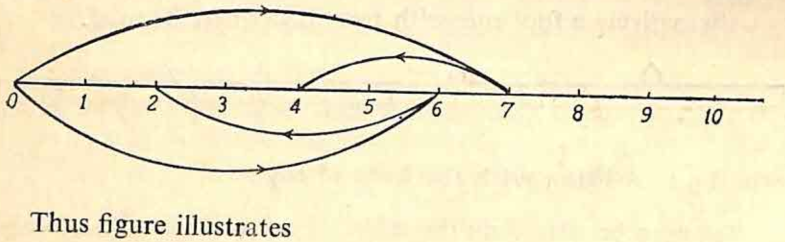
A large number of such examples can be constructed.

Experiment 3 : Some Methods for Teaching Subtraction

Some of the alternative methods available for this purpose are the following :

Method 1 : Subtracting with the help of the number line

To subtract 3 from 7, we first move seven steps to the right on the number line and then 3 steps to the left.

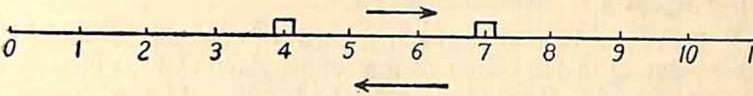


Thus figure illustrates

$$7-3 = 4$$
$$6-4 = 2$$

Arrows towards the right may be drawn in red and arrows towards the left in blue.

Method 2 : Subtraction with the help of Magnetic Counters

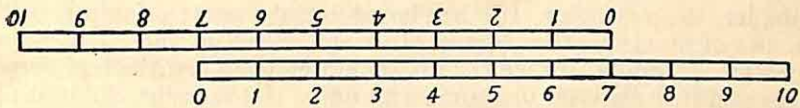


Move the red counter 7 steps to the right. Starting from here, move the blue counter 3 steps to the left to give the final position as 4 so that $7-3=4$.

Method 3 : Subtracting with the help of the ladder

Here to subtract 3 from 7, we first go 7 steps up and then come 3 steps down.

Method 4 : Subtracting with the help of foot-rules

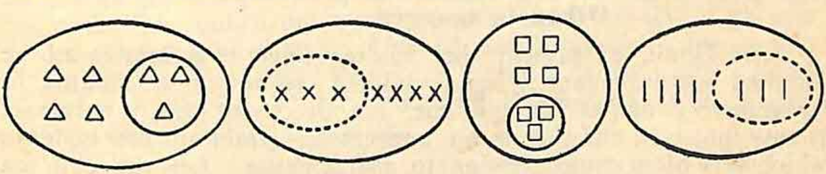


To subtract 3 from 7, invert the second foot rule and place it as above. The number on the original scale corresponding to 3 on the second scale gives $7-3=4$.

Method 5 : Subtraction with the help of toys

Put some toys on the table ; ask a child to choose any number of toys, say 7. Ask him to give 3 to someone and ask how many remain. Instead of toys, we may even use match box sticks or tooth-pricks etc.

Method 6. Subtraction with the help of symbols



Each of these figures represents $7-3=4$.

Experiment 4 : The Place Value System

For teaching this important concept, the first step is to teach the children how to write numbers in tens and units. For this purpose give 30 to 40 sticks (match box sticks or tooth-pricks) to each child and ask him to make bundles of ten each and bind these with flexible rubber bands and write the numbers as

Tens	Units
3	6

Each child may come to the board, write his number of tens and units, show the sticks to the class and say "I have got 3 tens and 6 sticks, *i.e.*, 36 sticks" and so on.

Experiment 5 : Number Board

There is a 10×10 number board of the shape *indicated in Experiment 23* in one corner of the room. Each child is given certain number of sticks. He makes them into bundles of tens, comes and places these bundles and single sticks on the appropriate square; the teacher checks. If it is all right, the child goes back to his seat, otherwise the child tries again.

Experiment 6 : New Names for Numbers in English

Let the children use names ten-one, ten-two, ten-three, ten-four, ten-five, ten-six, ten-seven, teng-enin, ten-nine, instead of the present names *viz.*, eleven, twelve, thirteen, fourteen, fifteen sixteen, seventeen, eighteen, nineteen respectively. These names (*i*) are simpler to remember, (*ii*) are based on the same principle as the names of numbers after twenty, (*iii*) are based on the place value system according to which we should indicate first the number of tens and then indicate the number of units, (*iv*) save the children the effort in remembering nine new words, thus reducing the labour of remembering twenty-eight words to that of remembering nineteen words only, (*v*) saves confusion which occurs now-a-days in spoken language between thirteen and thirty, fourteen and forty, etc., a confusion which arises because *e.g*,. thirteen and thirty are alternatives for three ten and three tens respectively.

Even otherwise the words eleven, twelve only remind us that at one time twelve was used as a base and the words, thirteen, fourteen etc. are unscientific since they place units first and ten afterwards.

Experiment 7 : New Names for Numbers in Hindi and Other Languages

In Hindi, at present the children have to remember all the hundred words separately and children spend three months in remembering names for ten one, ten two, ten nine, twenty one, twenty nine etc., and this is an unnecessary strain on the children which very often causes avasion to mathematics. Let the children learn the words 'dus ek', 'dus do', 'bees ek', 'bees do', bees nau', 'chalees nau' etc. in Hindi. This will mean reducing the labour for the child from that of remembering one hundred words to that of remembering only nineteen words, a substantial saving indeed !

Experiment 8 : Adding Number of Two Digits without Carrying

This can be done by using any of the above methods :

(*i*) number line (*ii*) ladder (*iii*) magnetic counters (*iv*) toys (*v*) symbols (*vi*) using the foot-rules (here for adding smaller numbers

use inches, for adding bigger numbers use centimetres and for adding still bigger numbers use metre sticks or get special rules made for each case).

Alternatively give each child two bundles of sticks. Thus suppose one child gets two bundles of 14 and 23 sticks. The child opens the first bundle, arranges it in one bundle of ten and gets 4 free sticks. Similarly he opens the second bundle, he gets two bundles of tens and three free sticks. He gets in all 3 bundles of tens and seven other sticks. He writes

Tens	Units
1	4
2	3
3	7

so that $14+23=37$.

Each child practises in the same way.

Experiment 9 : Adding through Addition of Tens

$$1 \text{ ten}+2 \text{ tens} = 3 \text{ tens} \quad \textit{i.e.,} \quad 10+20 = 30$$
$$3 \text{ tens}+4 \text{ tens} = 7 \text{ tens} \quad \textit{i.e.,} \quad 30+40 = 70$$
$$2 \text{ tens}+7 \text{ tens} = 9 \text{ tens} \quad \textit{i.e.,} \quad 20+70 = 90$$
$$14+23 = 1 \text{ ten}+4+2 \text{ tens}+3$$
$$= 3 \text{ tens}+7 = 30+7 = 37$$

or

$$14+23 = 10+4+20+3$$
$$= 30+7 = 37.$$

Experiment 10 : Adding with Carrying

Any of the methods of experiment 2 can be used. Alternatively to add 14 and 27, give 14 and 27 sticks. The child gets 1 bundle of ten and 4 free sticks from the first bundle and 2 bundles of ten and 7 free sticks from the second. Thus he gets 3 bundles of ten and 11 single sticks. From these eleven single sticks, he again gets 1 bundle of ten and one stick, so that the total number of sticks gives 4 bundles of ten and 1 single stick.

Tens	Units
1	4
2	7
4	1

Alternatively

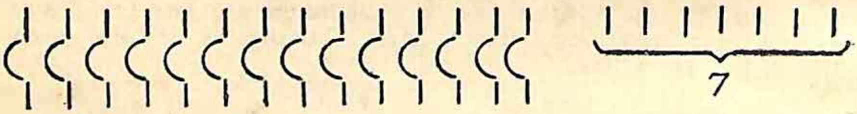
$$14+27 = 1 \text{ ten}+4+2 \text{ tens}+7 = 3 \text{ tens}+11$$
$$= 3 \text{ tens}+1 \text{ ten}+1 = 4 \text{ tens}+1$$
$$= 40+1 = 41$$

or

$$14+27 = 10+4+20+7 = 30+11 = 30+10+1$$
$$= 40+1 = 41.$$

Experiment 11 : Subtracting by Matching

To subtract 14 from 21, we take two bundles of 14 [and 21 sticks. Then we pair one stick of one bundle with one of the second and remove both. We continue till all the 14 of the second bundle are exhausted. We are left with 7 sticks from the other bundle. This gives $21-14=7$.



Experiment 12 : Subtraction without Borrowing

For subraction, we can use any of the methods given in experiment 3. We can use also the concept of tens and units. Thus to subtract 11 from 23 we write

$23=2$ tens$+3$

$11=1$ ten $+1$.

On subtracting, we get 1 ten$+2=12$. We can also do this as follows :

$$23-11 = (2 \text{ tens}+3)-(1 \text{ ten}+1) = (2 \text{ tens}-1 \text{ ten})+(3-1)$$
$$= 1 \text{ ten}+2 = 10+2 = 12$$

or $\quad 23-11 = (20+3)-(10+1) = (20-10)+(3-1)=10+2=12.$

For the second approach, we have to introduce the concept of parenthesis first, if the child does not already know it. From the earlier experiments, he already knows that we subtract tens from tens and units from units.

Experiment 13 : Subtraction with Borrowing

To subtract 13 from 21, we break these up into 1 ten$+3$ and 2 tens$+1$ respectively. We cannot subtract 3 from 1 and so we break up the second bundle as 1 ten$+1$ ten$+1$ $=$ 1 ten$+11$. Now we can subtract 3 from 11 to get 8 and subtracting 1 ten from 1 ten we get 0 ten.

$$21-13 = (2 \text{ tens}+1)-(1 \text{ ten}+3) = (1 \text{ ten}+1 \text{ ten}+1)-(1 \text{ ten}+3)$$
$$= (1 \text{ ten}+11)-(1 \text{ ten}+3) = (1 \text{ ten}-1 \text{ ten})+(11-3)$$
$$= 0 \text{ ten}+8 = 8.$$

Experiment 14 : Different Names for Numbers

We can write 4 in many ways, *e.g.*,

$$4 = 1+3 = 2+2 = 3+1 = 5-1 = 6-2 = 7-3 = 8-4$$
$$= 9-5$$

Let the children speak similar expressions for 0, 1, 5, 9, 11, etc. in as many different ways as possible.

Now 9, $1+8$, $2+7$, $3+6$, $3+3+3$, $10-1$, $11-2$ etc. are all different names for the same number.

The concept of number is an abstract concept. We use symbols 9, IX, ९, ||||||||| etc. to represent the same number.

The written forms are called numerals. What we write is a numeral, what we think is a number. Numeral is a symbol for a number which is a concept.

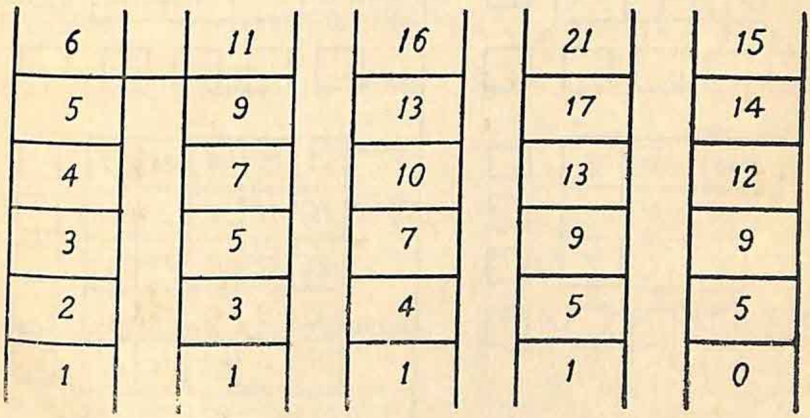
Experiment 15 : Use of 'Renaming' instead of 'Carrying' and 'Borrowing'

Let children add 58 and 34. Then first add units 8 and 4. Another name for $8+4$ is $10+2$. We give this name. Another name for $50+30+10$ is 90. We give this name and find that another name for $58+34$ is 92. Thus the problem of finding $58+34$ is that of finding name for this number which is 'single'. Thus $58+34 = 50+8+30+4 = 50+30+10+2 = 90+2 = 92$. We are not 'carrying', we are just 'renaming'.

Similarly let the children find $54-38$. We can rename 54 as $40+10+4$ or $40+8+6$ and rename 38 as $30+8$ and then carry out the subtraction process. The children can thus avoid the terms 'borrowing'. Thus, 'carrying' and 'borrowing' can both be avoided by using a more general term 'renaming'.

Most problems in mathematics, even at the highest level are of 'renaming', and this process makes mathematics a very useful science.

Experiment 16 : The Game of 5 Ladders



Ask a child to subtract 5 from 6, 4 from 5, 3 from 4, 2 from 3 and 1 from 2 in the first ladder. If he is successful, he goes to the

second ladder and subtracts in the same way. If he succeeds, he goes to the third ladder and so on.

Next ask a child to add all the elements of the first ladder. If he succeeds, he goes to the second ladder and adds all the elements.

In this way he continues till he completes both addition and subtraction from all ladders. The game continues till each child has come down and gone up each ladder.

Experiment 17 : Addition in Magic Squares

Total each row, each column and each diagonal (See figure below).

Ask the children to form more magic squares from a given magic square by :

(*i*) adding some number to each cell;

(*ii*) subtracting some number from each cell ;

(*iii*) multiplying the number in each cell by a given number ;

(*iv*) dividing the number in each cell by the same number ;

(*v*) interchanging first and third row or first and third columns in a 3×3 square ; and

(*vi*) rotating square about its centre.

4	3	8
9	5	1
2	7	6

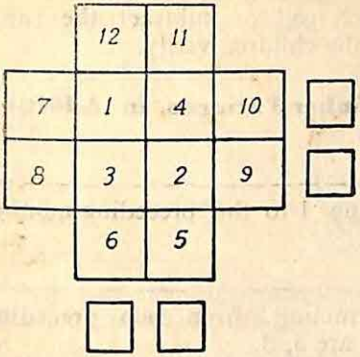
5	9	10
13	8	3
6	7	11

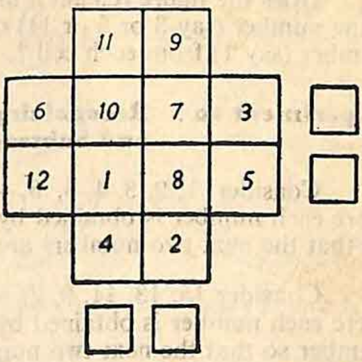
5	8	5
6	6	6
7	4	7

11	20	4	23	17
18	12	6	5	24
25	19	13	17	1
2	21	20	14	8
9	3	22	16	15

Experiment 18 : Addition in Magic Double Crosses

Examples are :

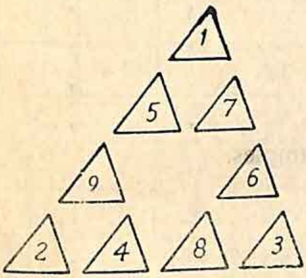


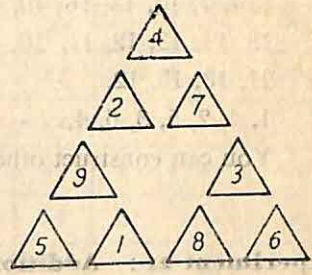


Ask the children to construct new magic double crosses either directly or by interchanging two rows or two columns.

Experiment 19: Addition in Magic Triangles and Hexagons

Examples are :





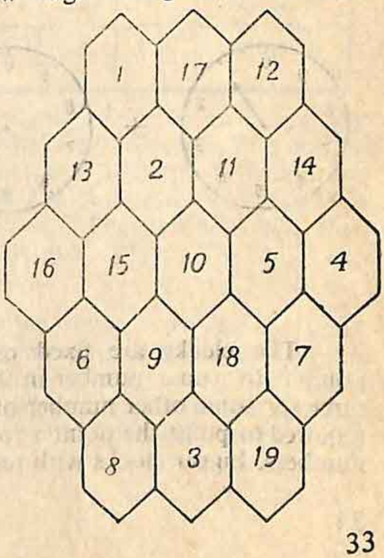
Ask the children to construct new magic triangles. An example of a magic hexagon is the following :

Each number from 1 to 19 has been used once only

The sum of every row containing 3 hexagons is 30

The sum of every row containing 4 hexagons is 40

The sum of every row containing 5 hexagons is 50



Let the children verify these facts and construct such examples.

Does the figure remain a magic hexagon if the children add the same number (say 3 or 5 or 11) to each cell or subtract the same number (say 1) from each cell ? Let the children verify.

Experiment 20 : Recognizing Number Patterns in Addition and Subtraction

Consider 1, 2, 3, 4, 5, 6, ———————————————.
Here each number is obtained by adding 1 to the preceding number so that the next two numbers are 7, 8.

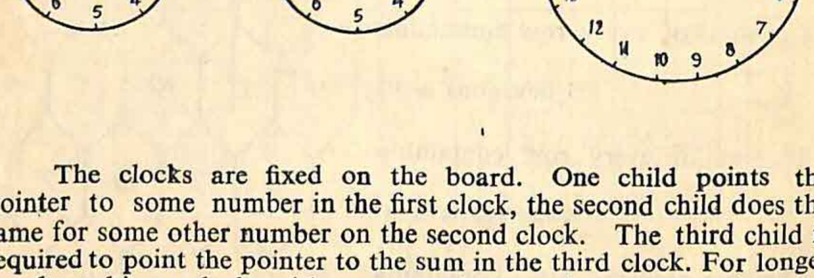
Consider 15, 13, 11, 9, 7, ——————————————.
Here each number is obtained by subtracting 2 from each preceding number so that the next two numbers are 5, 3.

In this way, ask the children to write the next three numbers in each of the following sequences :

1, 3, 5, 7, 9, 11, 13, —————, , ,
1, 4, 7, 10, 13, 16, 19, ————, , ,
15, 14, 13, 12, 11, 10, ————. , ,
21, 18, 15, 12, ————, , ,
1, 4, 2, 5, 3, 6, 4, —————, , ,

You can construct other similar examples.

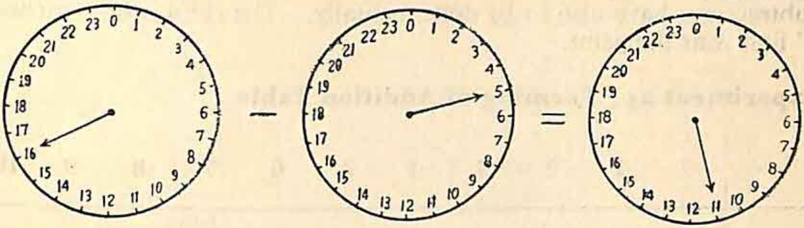
Experiment 21 : Addition Clocks



The clocks are fixed on the board. One child points the pointer to some number in the first clock, the second child does the same for some other number on the second clock. The third child is required to point the pointer to the sum in the third clock. For longer numbers, bigger clocks with more numbers can be constructed.

Experiment 22 : Subtraction Clocks

The same as in experiment 21, but the second child points to a number smaller than the first. The third child can mentally count



on the first clock how many steps it has to go backwards to reach the number on the second clock.

Experiment 23 : Variations of Snakes and Ladders

100	99	98	97	96	95	94	93	92	91
81	82	83	84	85	86	87	88	89	90
80	79	78	77	76	75	74	73	72	71
61	62	63	64	65	66	67	68	69	70
60	59	58	57	56	55	54	53	52	51
41	42	43	44	45	46	47	48	49	50
40	39	38	37	36	35	34	33	32	31
21	22	23	24	25	26	27	28	29	30
20	19	18	17	16	15	14	13	12	11
1	2	3	4	5	6	7	8	9	10

The game can be played on a horizontal floor or on a vertical iron plate with magnetic counters of different colours. The teacher throws a dice and asks the first child to move without touching the board at the intermediate points. If the child makes a mistake in addition, he remains in his earlier position. The second child then moves his counter. Five or six children can play. The child who reaches the top first wins.

Instead of using one dice, the teacher can use two and the child may be allowed to move a number of steps equal to the sum of the number of points. This will teach the child how to add mentally numbers up to 12. Alternatively we may use tickets numbered 0, 1, 2, 3, ...9. The teacher draws one at a time and the child moves the corresponding number of steps. The teacher replaces the tickets before drawing the ticket for the next child.

The game of snakes and ladders can also be played in reverse for subtraction.

For this purpose, children will start at 100 and move downwards. Subtractions have also to be done mentally. The child who reaches '1' first wins the game.

Experiment 24 : Forming of Addition Table

+	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10	11
2	2	3	4	5	6	7	8	9	10	11	12
3	3	4	5	6	7	8	9	10	11	12	13
4	4	5	6	7	8	9	10	11	12	13	14
5	5	6	7	8	9	10	11	12	13	14	15
6	6	7	8	9	10	11	12	13	14	15	16
7	7	8	9	10	11	12	13	14	15	16	17
8	8	9	10	11	12	13	14	15	16	17	18
9	9	10	11	12	13	14	15	16	17	18	19
10	10	11	12	13	14	15	16	17	18	19	20

For addition of 2 and 5, we move alo the row of 2 and th move down along the column of 5. We get the sum at the intersection. The teacher may draw tu lines t he board and ask eac chil d to complete a ro w or a co lmn.

The table can be used for quick addition of numbers remembering of addition facts, and to get partitions of numbers, ke

$16=6+10=7+9=8+8=9+7=10+6.$

The children can also make similar addition tables for numbers from 11 to 20 or 21 to 30 etc.

Experiment 25 : Practice in Addition Process; Reversed Addition Process

Usually children are given a large number of disconnected problems for addition. The children may find it interesting to do a single chain of problems in which they themselves form new problems and solve them. The following illustrates one possibility.

Reversed Addition Process :

Give the child a 4-digit number. Ask him to reverse the digits and add the number thus obtained to the original number. If the sum is a number of more than 4 digits, ask him to omit the unit digit. Again let him reverse the digits and add the two numbers. He may

continue the process eight or ten times and he may discover interesting patterns in this process *e.g.*,

1020	1234
+0201	+4321
1221	5555
+1221	+5555
2442	1111
+2442	+1111
4884	2222
+4884	+2222
9768	4444
+8679	+4444
18447	8888
+4481	+8888
6325	1777
+5236	+7771
11561	9548
+6511	+8459
7667	1800
+7667	+0081
15334	1881

The children will find many times numbers which do not change by reversing. Many times they would find cycles *e.g.*.

$$\begin{array}{r} 88 \\ +88 \\ \hline 176 \\ +71 \\ \hline 88 \\ +88 \\ \hline 176 \\ +71 \\ \hline 88 \end{array}$$

The children may be asked to find all two-digit numbers which have cycles, all three digit numbers with '1' in the unit place which

have cycles and so on. They may like the fun and in this process get a lot of practice of addition.

A modified reversed addition process can be defined in which when we get, on addition, five digit numbers we neglect the digit on the left. The children can find cycles for this process also.

Another interesting version arises when a child starts with a two-digit number, reverses it and adds it to the original number and continues the process till he gets a 'symmetrical' number *i.e.* a number whose digits from left and right are same. He will always succeed.

Different children can start with different two digit numbers and it can be a project for the whole class to find the number of steps required to reach a symmetrical number for all numbers from 10 to 99.

Experiment 26 : Reversed Subtraction Process :

In this process, we take a number of a certain specified number of digits We reverse the digits and subtract the smaller number from the larger. If the number obtained after subtraction is of smaller number of digits, we add zeros on the left and get the same number of digits as before.

We now reverse the digits and proceed as before *e.g.*,

693	792	594	990	891
−396	−297	−495	−099	−198
297	495	099	891	693

The number 693 is repeated and we get a cycle of 5 numbers.

(693, 297, 495, 099, 891)

Some other cycles the children will get are :

(09, 81, 63, 27, 45)
(396, 297, 495. 099, 891)
(36, 27, 45, 09, 81, 63)
(6993, 2997, 4995, 9990, 8999)
(6534, 2178)

Experiment 27 : Combined Reversed Subtraction and Addition Processes

For three digit numbers, the children would always get 000 or 1089, while for four digit numbers, they would get 0000 or 10980 or 9999. The children enjoy the existence of such patterns in three or four or five digit numbers.

Experiment 28 : Other Chain Processes

(*i*) You may ask the children to choose a two digit number, reverse and add and continue the process, without neglecting any digit, till he gets a five digit number.

(*ii*) Similarly the child may start with a four-digit number, continue the process of reversed subtraction without adding any zeros at any stage, till he gets 0 or a cycle.

(*iii*) A simpler process is obtained if the child starts with a number and subtracts at each stage only the sum of the digits in the number.

(*iv*) In this process a child chooses a number and subtracts from it a number obtained from it by finding the sum of digits of the original number and adding to it on the right as many zeros as possible subject to the condition that the number is smaller than the original number. The process is continued till the child gets zero.

Experiment 29 : Alternative Methods for Addition ; Left Hand Addition

(*a*) Sometimes the children forget "carrying" in addition. The following method reduces carrying to a minimum. To find the sum of given numbers, we add separately the digits in the unit place, in the tenth place etc. and write these sums in vertical columns, always shifting the sum one place to the left, then add the numbers in the column. Thus to find 3456+5678+2340+1456, we write

$$\begin{array}{lr} 6+8+0+6 = & 20 \\ 5+7+4+5 = & 21 \\ 4+6+3+4 = & 17 \\ 3+5+2+1 = & 11 \\ \hline & 12930 \\ \hline \end{array}$$

Let the children do some problems by both the usual method and this method and compare. Let them discover why the methods would always give the same result.

(*b*) An alternative method known as left hand addition is used in some new projects. This can completely eliminate carrying over. We add thousands, hundreds, tens and units separately and repeat the process, if necessary *e.g.*,

$$\begin{array}{rl} 2\,3\,5\,6 & \\ 5\,7\,8\,8 & \\ 8\,9\,6\,5 & \\ \hline 1\,5\,0\,0\,0 & \text{(15 thousands)} \\ 1\,9\,0\,0 & \text{(19 hundreds)} \\ 1\,9\,0 & \text{(19 tens)} \\ 1\,9 & \text{(19 units)} \\ \hline 1\,6\,0\,0\,0 & \text{(16 thousands)} \\ 1\,0\,0\,0 & \text{(10 hundreds)} \\ 1\,0\,0 & \text{(10 tens)} \\ 9 & \text{(9 units)} \\ \hline 1\,7\,1\,0\,9 & \end{array}$$

This takes a longer time, but is useful since it emphasizes basic principles. Let the children do some problems by this method.

(*c*) We write numbers from left to right and add numbers from right to left. In other words, we write first thousands, then hundreds, then tens and then units and in adding, we first add units, then tens, then hundreds and then thousands. This is anomalous. In 'forward arithmetic' or 'Arabic arithmetic', we write also numbers from right to left. Thus *e.g.*, we write two thousand three hundred and fifty six as 6532.

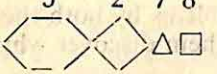
In this notation, the above sum will read as

```
      6 5 3 2
      8 8 7 5
      5 6 9 8
  ___________   In this notation no carrying is required.
  0 0 0 5 1
  0 0 9 1
  0 9 1
  9 1
  ___________
  9 0 1 7 1
```

Experiment 30 : An Alternative Method for Subtraction

This method which regards subtraction as complementary addition is very often preferred to the usual method of subtraction. To subtract a given number from another, we ask the question : What number should be added to the smaller of these numbers to get the bigger number ? Thus to find 5467 − 3278, we write

3 2 7 8



5 4 6 7

```
(1)(1)
3 2 7 8
2 1 8 9
_______
5 4 6 7
```

What should we add to get 7 or 17 ? We write 9 and carry '1' over. Next what should we add to (7 + 1) to get 6 or 16 ? We write 8 and carry over 1. What should we add to (2 + 1) to get 4 ? We write '1'. Similarly in the thousandth place, we have to add 2 to 3 to get 5. Thus 2189 is the answer.

The advantage of this method is that children have to remember only the addition facts. Subtraction facts have not to be remembered.

Shopkeepers very often use this principle *e.g.*, suppose you have bought goods worth Rs. 7 and 89 paise and you give the shopkeeper a ten-rupee note. He has to subtract Rs. 7·89 from Rs. 10·00 and return this amount. What he does is as follows :

He gives 1 paisa and says, this makes 90 paise. Then he gives 10 paise coin and says it makes one rupee and seven rupees were already there, that makes 8 rupees. He gives you two rupees more to make up Rs. 10.

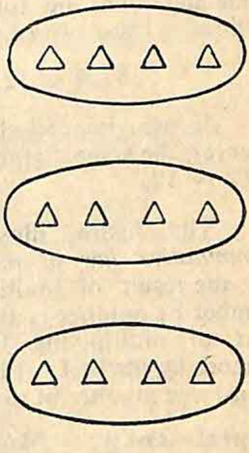
The children can actually carry out some monetary transactions. The teacher can act as the purchaser.

4

Multiplication and Division Operations

[EXPERIMENTS 31–45]

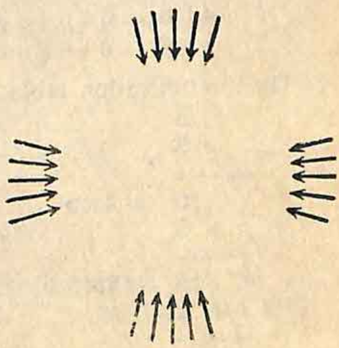
Experiment 31 : Multiplication by Equal Groups



Here are 3 sets of 4 objects each.

The total number of objects is 12.

$3 \times 4 = 12$



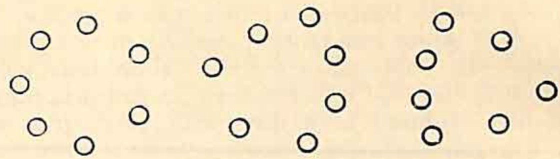
Here are 4 groups of 5 objects each

The total number is 20

$4 \times 5 = 20$

Here are 3 necklaces with 7 beads each. The total number of beads is 21.

$3 \times 7 = 21$

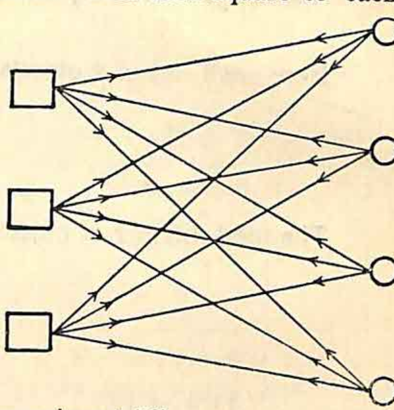


A large number of similar examples of toys on the teacher's table, counters on the board, symbols on the board, (★, □, ◇, △, ⬡ etc.) and sticks, can be constructed.

Experiment 32 : Multiplication by Correspondences

There are 3 squares and 4 circles. Join each square to each circle and count the total number of lines. These are 12.

$$3 \times 4 = 12$$



If we join circles to the squares, the same figure illustrates $4 \times 3 = 12$.

The figure illustrates the *commutative law of multiplication i.e.*, the result of multiplying one number by another is the same as that of multiplying the second number by the first. Illustrate this by a large number of examples (*c. f.* experiment 53).

Experiment 33 : Multiplication by Repeated Addition

$$3 \times 6 = 6+6+6 = (6+6)+6 = 12+6 = 18$$
$$4 \times 6 = 6+6+6+6 = (12+6)+6 = 18+6 = 24$$
$$5 \times 6 = 6+6+6+6+6 = (4 \times 6)+6 = 24+6 = 30$$

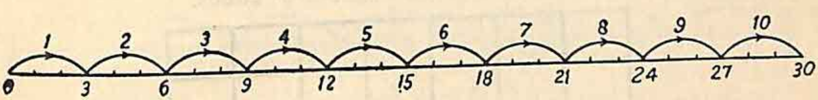
The multiplication table of 5 can be obtained as follows :

$$\begin{array}{rl} 5 & \\ +\ 5 & \\ \hline 10 & = 2 \times 5 \\ +\ 5 & \\ \hline 15 & = 3 \times 5 \\ +\ 5 & \\ \hline \end{array}$$

$$
\begin{array}{rl}
20 & = 4\times 5 \\
+\ 5 & \\
\hline
25 & = 5\times 5 \\
+\ 5 & \\
\hline
30 & = 6\times 5 \\
+\ 5 & \\
\hline
35 & = 7\times 5 \\
+\ 5 & \\
\hline
40 & = 8\times 5 \\
+\ 5 & \\
\hline
45 & = 9\times 5 \\
+\ 5 & \\
\hline
50 & = 10\times 5
\end{array}
$$

The children should be asked to obtain multiplication tables of 2 to 10 in the same way.

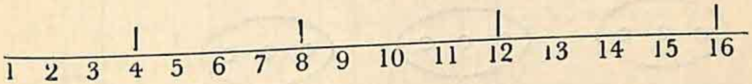
Experiment 34 : Multiplication by using the Number Line



Make groups of (say) 3 on the number line and read as

$1\times 3 = 3$, $2\times 3 = 6$, $3\times 3 = 9$, $4\times 3 = 12$, $5\times 3 = 15$, $6\times 3 = 18$, $7\times 3 = 21$, $8\times 3 = 24$, $9\times 3 = 27$, $10\times 3 = 30$.

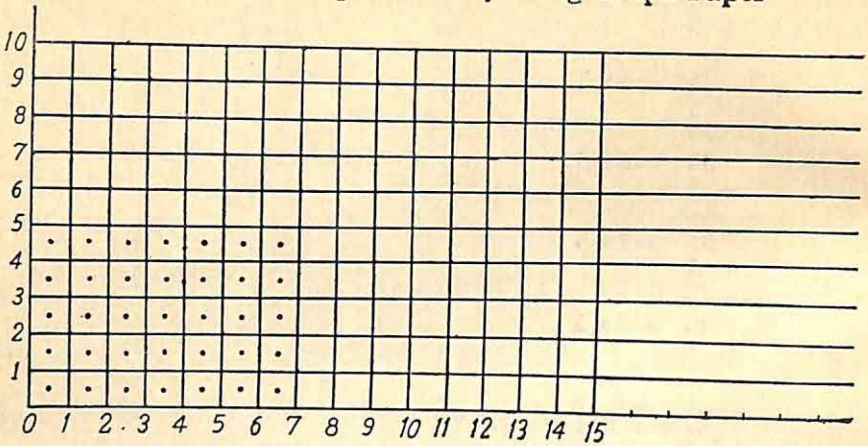
The children should be asked to obtain other multiplication tables in this way. They can obtain all these in one big chart. A metre stick with centimetre marks can be used on which chalk marks can be made.



The child counts 1, 2, 3, 4, puts a mark.

He again counts 1, 2, 3, 4 and puts another mark and so on. Later on he reads the multiplication table of 4 on the metre stick. Similarly all the tables can be prepared.

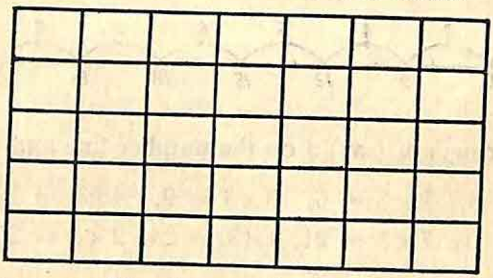
Experiment 35 : Multiplication by using Graph Paper



To multiply 7 by 5, count 7 steps on one side and 5 on the other and count the total number of squares. The children can prepare all the multiplication tables with the help of graph paper.

Experiment 36 : Multiplication by using Wooden Cubes

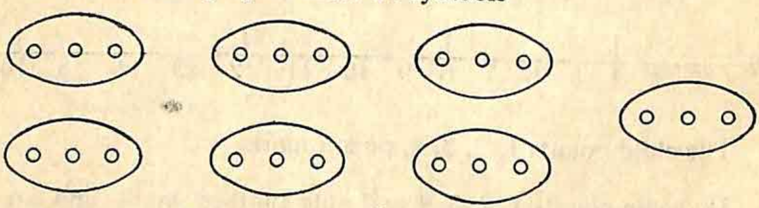
To multiply 7 by 5, put 7 wooden cubes on one side and put 5 on the other. And go on putting more cubes, till the rectangular box is completed. Count the total number of cubes.



Repeat the same with other products. With 100 cubes, all the multiplication tables upto 10×10 can be prepared.

Experiment 37 : Division by Equal Groups

To divide 21 by 3, we draw 21 symbols



and make groups of three each. We get 7 groups, so that

$$21 \div 3 = 7$$

Different symbols △, □, ★, ◇, ⬡ may be used and a large number of examples can be constructed.

Experiment 38. Division as Inverse of Multiplication

To divide 21 by 3, we repeat the multiplication table and find what number multiplied by 3 gives 21. This number is 7.

$3\times7=21$, $21\div3=7$, $21\div7=3$
$4\times6=24$, $24\div4=6$, $24\div6=4$
$7\times9=63$, $63\div7=9$, $63\div9=7$.

Give each child a multiplication fact and ask the child to deduce two division facts from this.

What about 0×7 ? It is $0+0+0+0+0+0+0=0$.

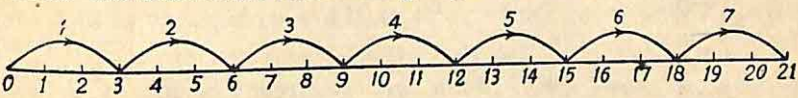
$\therefore \quad 0\times7=0$.

From here we can say $\frac{0}{7}=0$, but we cannot say $\frac{0}{0}=7$.

Since $0\times8=0$ and therefore $\frac{0}{0}=8$, so that $\frac{0}{0}$ would come out to be 8 also, but this is not possible. *Therefore division by 0 is not allowed in arithmetic.*

Experiment 39 : Division by using the umber Line

To divide 21 by 3, we mark off numbers up to 21 on the number line and divide these into groups of 3 each

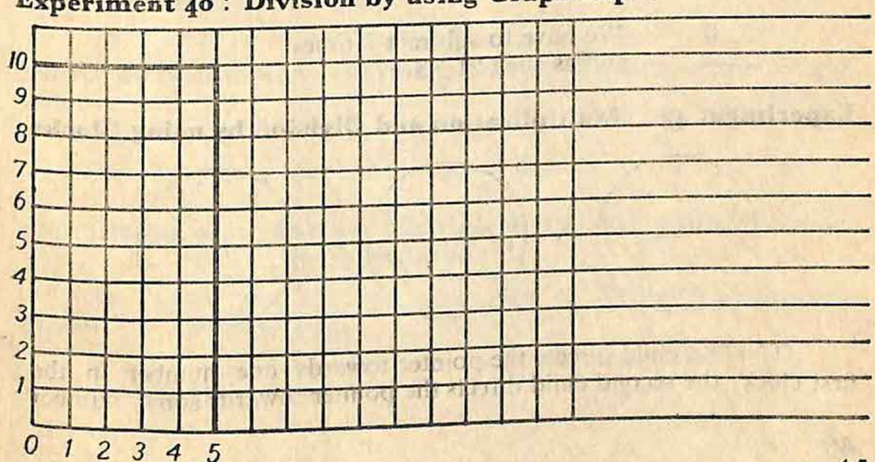


We get 7 groups. Therefore $21\div3=7$.

Repeat this with a number of similar problems.

If we are asked to divide 22 by 3, after 7 groups, one number is left over. We say $22\div3>7$; also $22\div3<8$. Ask the children to write similar inequalities for $48\div5<\square$, $48\div5>\square$, $29\div3<\square$, $29\div3>\square$ (*c.f.* experiment 65).

Experiment 40 : Division by using Graph Paper



To divide 50 by 5, we count off 5 marks horizontally and draw the vertical line through 5, then we begin counting upwards the small squares within *this vertical* line, till we reach 50, then we draw the horizontal line. Count on the vertical line. It gives therefore 10.

$50 \div 5 = 10.$

Experiment 41 : Division by using Counters or Cubes

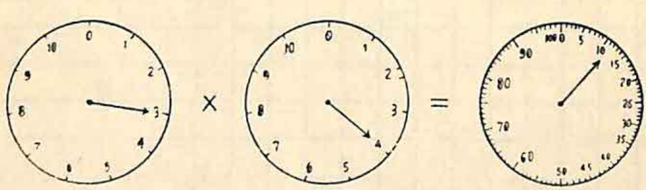
Suppose we have to find $49 \div 7$. We take 49 cubes and ask a child to divide these into groups of 7 each. The number of groups he gets gives the answer.

Experiment 42 : Division by Repeated Subtraction

$$\begin{array}{r} 21 \\ -3 \\ \hline 18 \\ -3 \\ \hline 15 \\ -3 \\ \hline 12 \\ -3 \\ \hline 9 \\ -3 \\ \hline 6 \\ -3 \\ \hline 3 \\ -3 \\ \hline 0 \\ \hline \end{array}$$

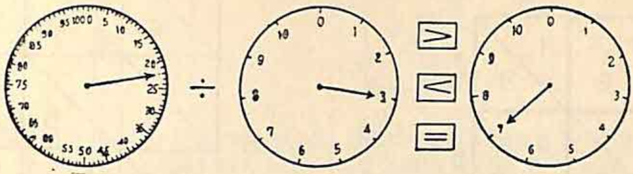
We have to subtract 7 times and as such $21 \div 3 = 7$.

Experiment 43 : Multiplication and Division by using Clocks



The first child directs the pointer towards one number in the first clock, the second child directs the pointer towards some number

in the second clock and the third child then points the pointer in the product clock towards the correct product. For division the teacher



directs the pointer to some number in the first clock, a child directs the pointer to a number in the second clock and a second child gives the answer in the third clock.

If the first clock shows 22 and the second shows 3, then the second child will show > 7 or < 8.

Experiment 44 : Division as an Operation*

Consider the set of natural numbers. Is it closed for division *i.e.* if we divide a natural number by another natural number, do we always get a natural number ? If we divide 22 by 7, we do not get a natural number. Therefore the set of natural numbers is not closed for division. Can we have a larger set, which includes the set of natural numbers but which is closed for division ? The answer is 'yes' and as the children will learn later, this larger set is the set of positive fractions.

Does the commutative law hold ? We know $6 \div 3 = 2$, but $3 \div 6$ is not even a natural number.

Does the associative law hold ?

$(12 \div 6) \div 2 = 2 \div 2 = 1$

$12 \div (6 \div 2) = 12 \div 3 = 4$

The associative law does not hold.

Does the distributive law hold ?

$18 \div (3+6) \neq (18 \div 3) + (18 \div 6)$,

but the other distributive law holds, whenever quotients are natural numbers.

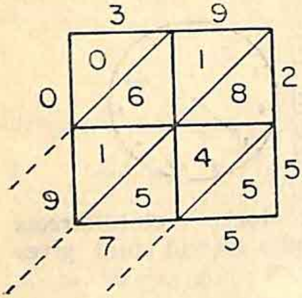
$(36+18) \div 9 = (36 \div 9) + (18 \div 9)$

This law is basic for the process of division.

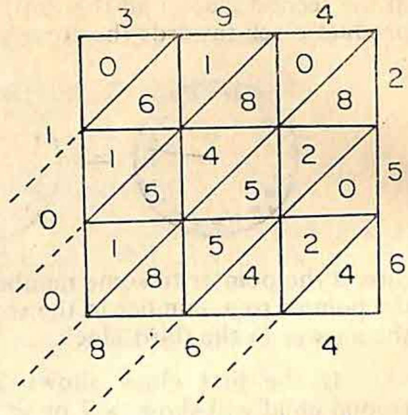
Experiment 45 : Lattice Multiplication and Multiplication by Napier Rods

The lattice multiplication procedure is illustrated in the two examples given on page 48.

* Experiment 44 may be done after the children have done experiments 53, 54, 55, 72 and 73.



$39 \times 25 = 975$

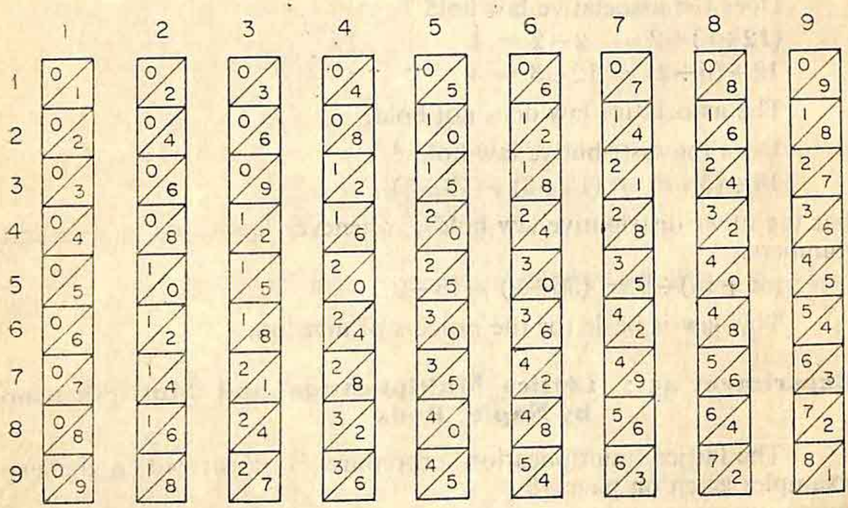


$394 \times 256 = 100864$

In the second example, we have to multiply 394 by 256. We write these numbers on the top and the right hand side of a 3×3 square as shown above. We write the products of the individual digits of the two numbers in the nine cells of the square, writing tens in the upper half and units in the lower half of each cell. We add numbers in slanting diagonal rows, carrying tens at each stage to the next diagonal. The product is then easily read.

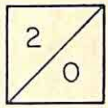
Let the children do a number of problems by this method.

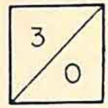
Multiplication by Napier's rods or apier's boxes is based on the principle. The children will enjoy preparing the following rods:



These rods actually give the multiplication table with the difference that units and tens of products have been separated.

To multiply 46 by 5, we look at the fifth cells of rod numbers 4 and 6 to find and we interpret it to get $46 \times 5 = 230$.





To multiply 394 by 256, we place third, ninth and fourth rod side by side and look at second, fifth and sixth cells of third rod. The procedure will then be similar to the procedure for lattice multiplication.

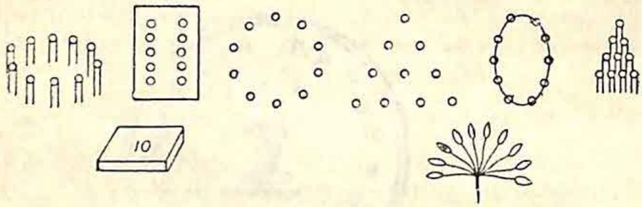
5

Numbers Greater than One Hundred

[EXPERIMENTS 46—50]

Experiment 46 : Introducing Numbers Greater than One Hundred

A 'ten' may be denoted by any one of the following



Ten such tens further give one hundred.

The children may be given two to three hundred sticks each. Each one has first to make bundles of ten each, then he can combine 10 bundles of these tens to get bundle of a hundred. Thus if the child has 328 sticks, he would make 32 bundles of ten, he would get 3 bundles of 100 each and 2 tens would remain. The child would report that he has 3 hundreds, 2 tens and eight units. Each child will come to board and write as below :—

Hundreds	Tens	Units
3	2	8
"	"	"

Which of the two children has got more sticks ? If one child has more hundreds than other, he has got more sticks than the other. If they have the same number of hundreds, then the child who has more tens has got more sticks. If they have same number of tens, then the child with more units has more. If they have the same number of units also, they have equal number of sticks (*c. f.* experiment 65).

Thus :

$3\ 2\ 8 > 2\ 9\ 9$

$3\ 7\ 6 > 3\ 6\ 8$

$4\ 2\ 5 > 4\ 2\ 1$

Discuss a large number of similar examples.

Experiment 47 : Adding Numbers Greater than One Hundred

In adding numbers, we add units separately, tens separately and hundreds separately, thus to find 328+411, we write

3 hundreds + 2 tens + 8
+4 hundreds + 1 ten + 1
7 hundreds + 3 tens + 9

or

or	3	2	8
+	4	1	1
	7	3	9

Ask the children to find 328+424

3 hundreds + 2 tens + 8
4 hundreds + 2 tens + 4
7 hundreds + 4 tens + 12

Now we convert 12 into one ten + 2, so that we get 7 hundreds +5 tens + 2 = 752

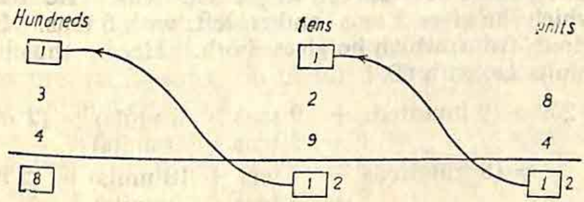
Similarly ask the children to add 328+494

3 hundreds + 2 tens + 8
4 hundreds + 9 tens + 4
7 hundreds + 11 tens + 12

We convert 12 into 1 ten + 2 units.

We get 12 tens. We convert these into one hundred and 2 tens. We get 8 hundreds + 2 tens and 2 units so that 328 + 494 = 822

We write this as



Explain clearly the principle of 'carrying over'. The bundles formed in the last experiment can be used for physical illustration of addition. Let the bundles of four children be combined together. Suppose the numbers are 328, 411, 424 and 494, then we get 15 bundles of hundreds, 14 bundles of tens and 17 units. We convert 17 units into 1 bundle of ten and 7 units. The 15 bundles of tens are converted

nto 1 bundle of hundred and 5 bundles of tens. Finally we get 16 bundles of hundreds, 5 bundles of tens and 7 units so that

$$328+411+424+494=1657$$

The 10 bundles of hundreds can be combined to give one bundle of a thousand. Thus we get one bundle of a thousand, 6 of hundreds, 5 of tens and 7 units.

Let each group of four children add the numbers of their sticks and write their results as :

Thousands	Hundreds	Tens	Units
1	6	5	7
,,	,,	,,	,,
,,	,,	,,	,,
,,	,,	,,	,,

Ask them to add these again as above.

Experiment 48 : Subtraction of Numbers Greater than One Hundred

Suppose a child has 328 sticks. You ask him to give 215. From his 8 units, he gives you 5 units and is left with 3 units. From his 2 tens, he gives 1 ten and is left with one ten. From his 3 hundreds, he gives 2 hundreds and is left with 1 hundred.

This is represented by

Hundreds	Tens	Units
3	2	8
−2	1	5
1	1	3

Now suppose you ask him to give you 259 sticks.

From his 8 units, he has to give 9 units. This is not possible. He opens out one bundle of ten and gets 18 units. He gives 9 and is left with 9. He is now left with 1 ten and he has to give 5 tens. He opens one bundle of hundred to get ten tens. He has now 11 tens from which he gives 5 tens and is left with 6 tens. He is left with 2 hundreds from which he gives both. He is thus left with 6 tens and 9 units *i.e.* with 69.

$$\begin{aligned}328-259 &= (3 \text{ hundreds} + 2 \text{ tens} + 8 \text{ units}) - (2 \text{ hundreds} + 5 \text{ tens} + 9 \text{ units})\\ &= (3 \text{ hundreds} + 1 \text{ ten} + 18 \text{ units}) - (2 \text{ hundreds} + 5 \text{ tens} + 9 \text{ units})\\ &= (3 \text{ hundreds} + 1 \text{ ten} + 9 \text{ units}) - (2 \text{ hundreds} + 5 \text{ tens})\\ &= (2 \text{ hundreds} + 11 \text{ tens} + 9 \text{ units}) - (2 \text{ hundreds} + 5 \text{ tens})\\ &= 6 \text{ tens} + 9 \text{ units} = 69\end{aligned}$$

or	Hundreds	Tens	Units
	3	2	8
−	2	5	9

is the same as	Hundreds	Tens	Units
	2	11	18
−	2	5	9
		6	9

A large number of such problems should be done and the process of borrowing or of 'conversion' of one hundred into 10 tens and 1 ten into 10 units should be clearly explained.

Experiment 49 : Addition and Subtraction with the Help of Currency Notes

Get some thousand-rupee, hundred-rupee, ten-rupee and one-rupee 'notes' from the market. These are available for a few paise only or you can make these in the class. The teacher can act as a banker who can exchange one thousand-rupee note for ten hundred-rupee notes or one hundred-rupee note for ten ten-rupee notes etc.

Give each child 3 sets of notes, each set consisting of a few (less than 10) notes of each kind. You may not give him a note of one kind at all. These notes are pinned together. The child goes to his seat and writes the number of notes of each kind in each set as follows :

	Thousands	Hundreds	Tens	Units
First set	...	...	...	...
Second set	...	...	...	...
Third set	...	...	...	...
Sum set	...	...	...	...

The child is asked to add these. He adds first one-rupee notes. If these exceed 10, he comes to the teacher, gives him 10 one-rupee notes and gets from him one ten-rupee note. He goes back to his seat and adds all ten-rupee notes. If these exceed 10, he again comes to the teacher and gets these exchanged into hundred-rupee notes. The procedure is repeated for hundred-rupee notes. After this he writes the sum and checks it.

After a child has done this sum, he is given another 3 sets of notes, he repeats the process. Afterwards he is asked to add the two answers.

For subtraction, pairs of children are formed. A is given one set of notes by the teacher. He has to give to B an amount specified by the teacher. B can also ask for an amount, but it has to be less than what A has got. A can exchange notes with the teacher to satisfy B's demand. Afterwards A's and B's role are interchanged.

If the child is given ten or more notes of one kind, he has to get these first converted into notes of higher denomination before starting the process of addition or subtraction.

Experiment 50 : Addition and Subtraction of Rupees and Paise

The procedure is the same as above. Use thousand rupee, hundred-rupee, ten-rupee and one-rupee notes and coins of ten-paisa and one paisa (paper-models). For adding, the child starts by adding one paisa coins. If these exceed 10, he gets ten-paisa coins from the teacher for each ten of these. He adds these ten-paisa coins. If these exceed 10, he gets each ten of these converted into a rupee-note and so on. To separate rupees and paise, use a point. Thus Rs. 328 and 95 paise are written as Rs. 328·95 and Rs. 249 and 7 paise are written as Rs. 249·07 (since there is no ten-paisa coin).

To subtract the second sum from the first, the child has first to give 7 paise, but he has got only 5 one-paisa coins, so he goes to the teacher, converts one of his ten-paisa coins into 10 one-paisa coins and pays 7 one-paisa coins out of the 15 he has. He is left with 8 one-paisa coins, and 8 ten-paisa coins. Since he has not to pay any ten-paisa coins, he keeps all these. Next he has to give 9 one-rupee notes, but he has only 8. So he goes to the teacher, gets one ten-rupee note converted, gets 18 one-rupee notes in all, pays 9 one-rupee notes and keeps 9 one-rupee notes. Now he has only 1 ten-rupee notes and he has to pay 4. So he gets, one hundred-rupee note converted into 10 ten-rupee notes. He gets 11 ten-rupee notes in all from which he pays 4 and keeps 7. Finally he is left with no hundred-rupee note, 7 ten-rupee notes, 9 one-rupee notes, 8 ten-paisa coins and 8 one-paisa coins *i.e.* with Rs. 79 and 88 paise. He writes it as Rs. 79·88 so that

$$
\begin{array}{r}
\text{Rs. } 328{\cdot}95 \\
-\text{Rs. } 249{\cdot}07 \\
\hline
\text{Rs. } 79{\cdot}88 \\
\hline
\end{array}
$$

After doing a few such sums of addition and subtraction, the child notices that except for the point, it is ordinary subtraction (and so it is). In this way addition and subtraction of decimals can be easily taught.

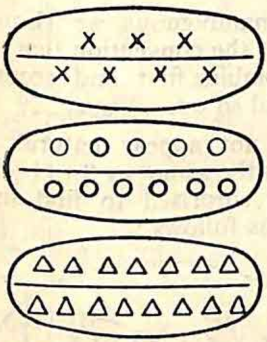
6

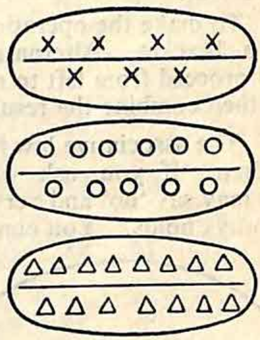
Laws of Arithmetic

[EXPERIMENTS 51—60]

Experiment 51 : Commutative Law (Change of Order Principle)

Ask the children to notice from the table that





$3+4=7$,
$5+6=11$,
$7+8=15$,

$4+3=7$
$6+5=11$
$8+7=15$

i.e. when we add two numbers, we get the same result, whether we add the first number to the second or the second to the first. This is called the *commutative law* and holds for addition composition.

It does not hold for subtraction, since

$$6-5=1, \qquad 5-6=-1$$

so that $\quad 6-5\neq 5-6,$

i.e., here it matters whether we subtract 5 from 6 or 6 from 5.

Let the child find $4+26$ on the number line and then $26+4$. Which is easier ? Point out the advantage of the use of the commutative law here.

Experiment 52 : The Associative Law (The Grouping Principle)

$(3+4)+5=7+5=12$; $3+(4+5)=3+9=12$
$(4+5)+2=9+2=11$; $4+(5+2)=4+7=11$.

Addition is a binary operation *i.e.*, we add two numbers at a time. What does $3+4+5$ mean ? It may mean two things. We may first add 3 and 4 to get a number and then add this to 5 to get the result or we may first add 4 and 5 and then add 3 to this to get the result, *i.e.*, we may associate 3 and 4 first or 4 and 5 first. Will the result always be the same ? The table shows that it is so. This fact is expressed by saying that the composition of addition is associative in the set of natural numbers (*c.f.* experiment 72). Every composition is not associative, *e.g.*, consider the operation of subtraction. What does $5-4-3$ mean?

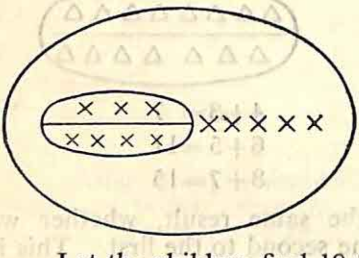
We may associate 5 and 4 first or 4 and 3 first. We get in the first case $(5-4)-3=1-3=-2$

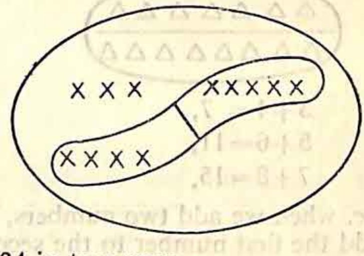
We get in the second case $5-(4-3)=5-1=4$

These two are different. *The associative law does not hold for subtraction.*

To make the operation of subtraction unambiguous, we should use a bracket. Alternatively we can adopt the convention that we shall proceed from left to right *i.e.* we first combine first and second and then combine the result with the third and so on.

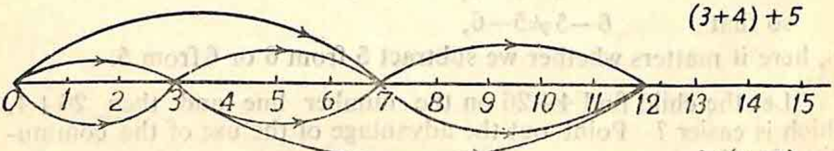
The associative law for addition does not appear natural to children. If you ask whether $(3+4)+5$ is the same as $3+(4+5)$, they may say 'no' and very often they will be surprised to find that it always holds. You can illustrate the law as follows :





Let the children find $19+196+204$ in two ways.

Which is easier ? The law can also be illustrated on the number line as follows :



Experiment 53 : Commutative and Associative Laws for Multiplication

Consider the set of positive integers. If we multiply a positive integer by a positive integer, we get a positive integer. The set of positive integers is thus *closed* for multiplication.

The child can be asked to prepare the following multiplication table.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

(1) From the table, let the child read 5×6, 6×5, 7×8, 8×7 etc. and verify the commutative law.

×	×	×	×	×	×
×	×	×	×	×	×
×	×	×	×	×	×
×	×	×	×	×	×
×	×	×	×	×	×

×	×	×	×	×	×	×	×
×	×	×	×	×	×	×	×
×	×	×	×	×	×	×	×
×	×	×	×	×	×	×	×
×	×	×	×	×	×	×	×
×	×	×	×	×	×	×	×
×	×	×	×	×	×	×	×

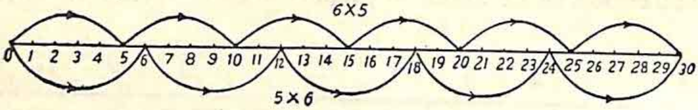
5 groups (rows) of 6 each
$=5\times 6$

or 6 groups (columns) of 5 each
$=6\times 5$

7 groups (or rows) of 8 each
$=7\times 8$

8 groups (columns) of 7 each
$=8\times 7$

The law can also be illustrated on the number line as follows :



(2) Find $(4\times 2)\times 3$ and $4\times(2\times 3)$
$(1\times 3)\times 3$ and $1\times(3\times 3)$
$(2\times 3)\times 2$ and $2\times(3\times 2)$

and verify in each case that the associative law holds *i.e.* whether we associate first and second numbers first or second and third numbers first, the result is the same.

Take 24 unit cubes and arrange these in a rectangular solid $4\times 2\times 3$. Looked at from one side there are 3 rows of 4×2 cubes. Looked at from another side there are 4 columns of 2×3 cubes so that

$$(4\times 2)\times 3=4\times(2\times 3)$$

(3) Note that multiplication tables are given both by rows and columns of the table.

(4) Note there is symmetry about one diagonal, shown by the dotted line.

(5) The cells on the diagonal give
$1\times 1=1$, $2\times 2=4$, $3\times 3=9$, $4\times 4=16$, $5\times 5=25$

These are written as :

$1^2=1$, $2^2=4$, $3^2=9$, $4^2=16$, $5^2=25$,
$6^2=36$, $7^2=49$, $8^2=64$, $9^2=81$, $10^2=100$.

Point out the scientific notation according to which
$7^3=7\times 7\times 7$, $7^4=7\times 7\times 7\times 7$

and how it depends on the associative law and how this notation saves space. Thus

$3^7=3\times 3\times 3\times 3\times 3\times 3\times 3$; and $3^7\neq 7^3$

The result is different whether first number is raised to the power equal to the second or second number is raised to the power equal to the first. The commutative law does not hold for this operation.

Experiment 54 : Combination of Commutative and Associative Laws

(*i*) Let the children find $197+3+296+4+195+5$ by associating numbers differently and remembering that addition is a binary operation, *i.e.*, only two numbers can be added at a time.

$(197+3)+[(296+4)+(195+5)]$
$= 200+[300+200] = 200+500=700$
$[(197+3)+(296+4)]+(195+5)$
$= (200+300)+200 = 500+200 = 700$
$[\{(197+3)+296\}+\{4+(195+5)\}]$
$= \{496+204\} = 700$

The sum is independent of the position of the brackets. This result is called the *generalised associative law for addition.*

(*ii*) Similarly according to the *generalised associative law for multiplication*, in an expression of the form $3\times4\times5\times6\times8\times11$, we can have associative numbers in any way we like so that

$3\times4\times5\times6\times8\times10 = 12\times5\times6\times8\times10$
$= 60\times6\times8\times10 = 360\times8\times10 = 2880\times10 = 28800$
$3\times4\times5\times6\times8\times10 = 12\times30\times80 = 360\times80 = 28800$
$3\times4\times5\times6\times8\times10 = (12\times5)\times(48\times10)=60\times480$
$= 28800$

Let the children verify the generalised associative and commutative laws for addition and multiplication by solving a number of examples.

(*iii*) In practice, we use commutative and associative law in combination, *i.e.*, we change order and associate numbers arbitrarily. Thus if we use the laws separately, we get

$4+5+6+194+295+396$
$= 4+5+200+295+396$ (using associative law)
$= 4+200+5+295+396$ (using commutative law)
$= 200+4+295+5+396$ (using commutative law)
$= 200+4+300+396$ (using associative law)
$= 200+300+4+396$ (using commutative law)
$= 500+400$ (using associative law)
$= 900$

On the other hand by using the combined-associative-commutative law for addition, we can write

$4+5+6+194+295+396$
$= (396+4)+(295+5)+(194+6)$
$= 400 \quad +300 \quad +200 = 900$

In the same way,

$4\times3\times110\times225 = (225\times4)\times(110\times3)$
$= 900\times330 = 297000$

Let the children realise that addition and multiplication problems are considerably simplified by the use of combined-associative-commutative laws and the corresponding simplifications are not available for the operations of subtraction and division.

Experiment 55 : Distributive Law or Multiplication-Addition Principle.

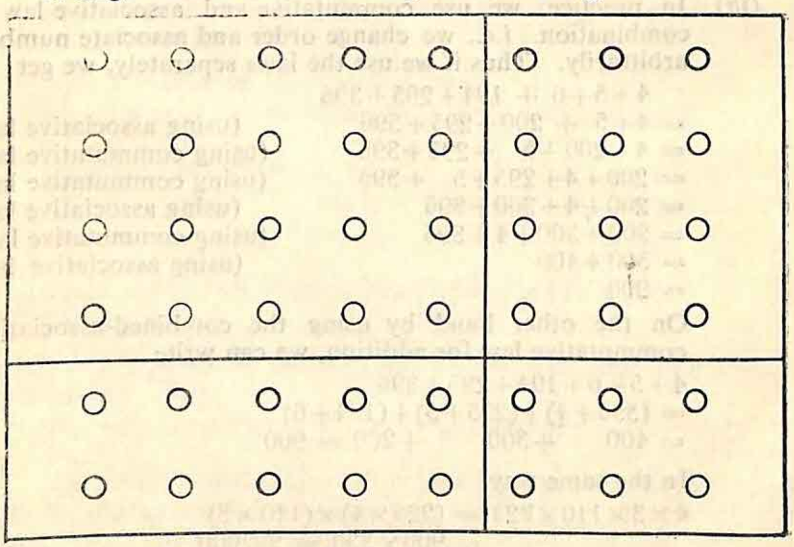
From the multiplication table, we verify

$2 \times (3+4) = 2 \times 7 = 14$	$(2 \times 3) + (2 \times 4) = 6 + 8 = 14$
$3 \times (4+5) = 3 \times 9 = 27$	$(3 \times 4) + (3 \times 5) = 12 + 15 = 27$
$2 \times (3+5) = 2 \times 8 = 16$	$(2 \times 3) + (2 \times 5) = 6 + 10 = 16$

The law illustrated by these examples is known as the *distributive law* and is useful in multiplication problems.

$$5\times(6+8) = 5\times\square+5\times\triangle$$
$$4\times(6+2) = 4\times\square+4\times\triangle$$
$$\begin{aligned}(3+2)\times(4+5) &= (3+2)\times 4+(3+2)\times 5\\ &= 4\times(3+2)+5\times(3+2)\\ &= (4\times 3)+(4\times 2)+(5\times 3)+(5\times 2)\end{aligned}$$

A large number of such examples should be given.



$$6\times(5+3) \qquad = 6\times 5+6\times 3$$

The bigger rectangle contains $6\times(5+3)$ objects, while the two rectangles contain 6×5 and 6×3 objects respectively.

The bigger rectangle is also broken into 4 smaller rectangles giving

$$(2+4)\times(5+3) = (2\times 5)+(4\times 5)+(2\times 3)+(4\times 3)$$

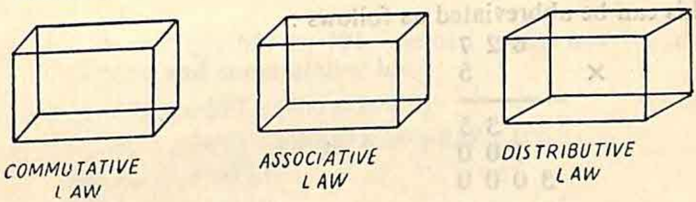
Draw many other similar illustrations for the distributive law.

Experiment 56 : Recognition of Laws

Write on some cards equations like the following :

$$10+6 = 6+10$$
$$5\times 4 = 4\times 5$$
$$5\times(4\times 6) = (5\times 4)\times 6$$
$$5+(5+6) = (5+5)+6$$
$$5\times(6+5) = (5\times 6)+(5\times 5)$$
$$(27+18)\div 3 = (27\div 3)+(18\div 3)$$

Put three boxes marked associative law, distributive law and commutative law and ask each child to drop the cards into the correct box. His score is the correct number of cards dropped.



Experiment 57 : Multiplication through the Use of the Distributive Law

The object of this experiment is to understand the process of multiplication of natural numbers.

To multiply 627 by 5 we can proceed by two methods :

Method 1 : By repeated addition

Hundreds		Tens		Units
1		3		
6		2		7
6		2		7
6		2		7
6		2		7
6		2		7
31	1	3	3	5

We add 7 five times (*i.e.* multiply 7×5) to get 35. We carry over 3 tens. We add 2 tens five times and add 3 tens to get 13 tens. We carry over 10 tens as one hundred and add 6 hundreds five times to get 31 hundreds.

This process is abbreviated as follows :

$$\begin{array}{r} 6\,2\,7 \\ \times \quad\quad 5 \\ \hline 3\,1\,3\,5 \end{array}$$

Let the children do some multiplication problems by repeated addition.

Method 2 : By using distributive law

$$\begin{aligned} (627\times5) &= (600+20+7)\times5 \\ 600\times5 &= 3000 \\ 20\times5 &= 100 \\ 7\times5 &= 35 \\ \hline 627\times5 &= 3135 \end{aligned}$$

This can be abbreviated as follows :

$$\begin{array}{r} 6\,2\,7 \\ \times \quad\quad 5 \\ \hline 3\,5 \\ 1\,0\,0 \\ 3\,0\,0\,0 \\ \hline 3\,1\,3\,5 \\ \hline \end{array}$$

Let the children do some multiplication problems by this method.

To multiply 627 by 15, we can proceed by either of the above methods, but we have to know the multiplication table for 15. This may be possible, but suppose we have to multiply by 125, we cannot remember multiplication table of 125 and so the first method is out of question. The second can be used as follows :

$$627\times125 = 627\times(100+20+5)$$

$$627\times100 = 62700$$

$627\times 20 = 12540$ (We need multiplication table of 2 only)

$627\times 5 = 3135$ (We need multiplication table of 5 only)

$$\begin{array}{r} \hline 78375 \\ \hline \end{array}$$

This can be abbreviated as follows :

$$\begin{array}{r} 6\ 2\ 7 \\ \times \quad 1\ 2\ 5 \\ \hline 6\ 2\ 7\ 0\ 0 \\ 1\ 2\ 5\ 4\ 0 \\ 3\ 1\ 3\ 5 \\ \hline 7\ 8\ 3\ 7\ 5 \\ \hline \end{array}$$

In the first row, we have multiplied by 100, in the second row, we have multiplied by 20, while in the third row, we have multiplied by 5 only.

In this process, we have to remember multiplication tables up o 9 only.

This method is slightly different from the usual multiplication method. Let the children do sufficient examples by this method.

Method 3 : To multiply 627 by 125, we can again use the distributive and commutative laws

$$\begin{aligned} (627)\times(125) &= 627\times(100+20+5) \\ &= 627\times5+627\times20+627\times100 \end{aligned}$$

$$\begin{array}{r} 627\times\ \ 5=\ \ 3135 \\ 627\times\ 20=12540 \\ 627\times100=62700 \\ \hline 78375 \\ \hline \end{array}$$

This can be abbreviated as follows :

$$\begin{array}{r} 627 \\ \times \quad 125 \\ \hline 3135 \\ 12540 \\ 62700 \\ \hline 78375 \\ \hline \end{array} \qquad \text{or,} \qquad \begin{array}{rrrrr} & & 6 & 2 & 7 \\ \times & & 1 & 2 & 5 \\ \hline & 3 & 1 & 3 & 5 \\ 1 & 2 & 5 & 4 & \\ 6 & 2 & 7 & & \\ \hline 7 & 8 & 3 & 7 & 5 \\ \hline \end{array}$$

This is the usual method of multiplication. Let the children do a number of problems from first principles using this method.

Method 4. We can use the double distributive law *e.g.*

$$\begin{aligned}627\times 125 &= (600+20+7)\times(100+20+5)\\ &= 600\times(100+20+5)+20\times(100+20+5)\\ &\qquad +\ 7\times(100+20+5)\\ &= 600\times 100+600\times 20+600\times 5\\ &\qquad +20\times 100+20\times 20+20\times 5+7\times 100\\ &\qquad +7\times 20+7\times 5\end{aligned}$$

$$\begin{aligned}600\times 100 &= 60000\\ 600\times 20 &= 12000\\ 600\times 5 &= 3000\\ 20\times 100 &= 2000\\ 20\times 20 &= 400\\ 20\times 5 &= 100\\ 7\times 100 &= 700\\ 7\times 20 &= 140\\ 7\times 5 &= 35\\ \hline &\ \ 78375\\ \hline\end{aligned}$$

Let the children do some problems by this method.

Experiment 58 : Division Through Repeated Subtraction and Division by the Use of the Distributive Law

To divide 54 by 3, we have to subtract multiples of three from 54. We have to find how many times we have to subtract three to get zero. We do not subtract threes one by one. We subtract as great a multiple of 3 as possible. It is convenient to subtract 10 or 20 or 30 threes. We subtract 10 threes to get remainder 24. We again subtract 8 threes to get zero. Thus 18 threes, in all, have to be subtracted to get zero, so that $54\div 3=18$.

All this can be represented as follows :

$$\begin{array}{rll} 18 & & \\ 3\overline{)54} & & \\ -30 & 10 & \text{threes} \\ \hline 24 & & 54\div 3=18 \\ -24 & 8 & \text{threes} \\ \hline 0 & & \end{array}$$

Similar examples are the following :

$$\begin{array}{rll} & 24 & \\ 4) & 96 & \\ & -40 & 10 \text{ fours} \\ & 56 & \\ & -40 & 10 \text{ fours} \\ & 16 & \\ & -16 & 4 \text{ fours} \\ & 0 & \end{array}$$

$$96 \div 4 = 24$$

$$126 \div 6 = 21$$

$$\begin{array}{rll} & 21 & \\ 6) & 126 & \\ & -60 & 10 \text{ sixes} \\ & 66 & \\ & -60 & 10 \text{ sixes} \\ & 6 & \\ & -6 & 1 \text{ six} \\ & 0 & \end{array}$$

These can be slightly simplified as follows :

$$\begin{array}{rll} & 24 & \\ 4) & 96 & \\ & -80 & 20 \text{ fours} \\ & 16 & \\ & -16 & 4 \text{ fours} \\ & 0 & \end{array}$$

$$\begin{array}{rll} & 21 & \\ 6) & 126 & \\ & -120 & 20 \text{ sixes} \\ & 6 & \\ & -6 & 1 \text{ six} \\ & 0 & \end{array}$$

If we try to divide 97 by 4, we would get '1' as *remainder* and 24 as *quotient*.

Now consider division of 1848 by 33. We consider the subtraction of 10×33 *i.e.*, 330 but that would be too little, so we try $20 \times 33 = 660$, $30 \times 33 = 990$, $40 \times 33 = 1320$, $50 \times 33 = 1650$, $60 \times 33 = 1980$. The last one becomes greater than 1848, so we subtract 50×33 and get 198 as remainder. Again we try to subtract 1×33, 2×33, 3×33, 4×33, 5×33, 6×33. We write this as

$$\begin{array}{rll} & 56 & \\ 33) & 1848 & \\ & -1650 & 50 \times 33 \\ & 198 & \\ & -198 & 6 \times 33 \\ & 0 & \end{array}$$

$$1848 \div 33 = 56$$

A large number of such examples can be constructed.

Another Method. For seeing how distributive law for division can be applied, consider the last example :

$$1848 \div 33 = (1650 + 198) \div 33 = 1650 \div 33 + 198 \div 33$$
$$= 50 + 6 = 56.$$

or $1848 \div 33 = (1320+330+198) \div 33 = 40+10+6 = 56$

The children can do a number of problems by this method. Each problem can be done in a large number of ways by breaking up the dividend in a number of different ways.

Experiment 59 : Division as a process of successive approximation

Successive approximations to get an answer are very important in mathematics *i.e.* we get the answer in a number of steps, each step taking us nearer the answer. Division process gives the first elementary example of this important process.

Consider $625 \div 5 = \square$ or $5 \times \square = 625$

now $5 \times 100 = 500 < 625$, $\quad 5 \times 200 = 1000 > 625$

$\therefore$ $\square$ lies between 100 and 200.

Subtract 500 from 625. We get 125.

$5 \times 20 = 100 < 125$, $\quad 5 \times 30 = 150 > 125$.

$\therefore$ $\square$ lies between 120 and 130

Find $625 - 5 \times 120 = 25 = 5 \times 5$.

Thus the answer is 115.

5×100	5×120	5×125	5×130	5×200
500	600	625	650	1000

Let the children do a number of simple division problems by this method, in which they come nearer to the answer from both sides.

Experiment 60 : Multiplication of decimal fractions by natural numbers

Suppose the child has to multiply 328·95 by 5. He can do so by adding 328·95 five times. Alternatively he can consider the problem of multiplying Rs. 328 and 95 paise by 5. He multiplies each separately to get Rs. 1640 and 475 paise. He converts the paise into Rs. 4 and 75 paise to get a total of Rs. 1644 and 75 paise or Rs. 1644·75 so that

$$\begin{array}{lr} \text{Rs.} & 328{\cdot}95 \\ \times & 5 \\ \hline \text{Rs.} & 1644{\cdot}75 \\ \hline \end{array}$$

Again it is noticed that except for the decimal point, the multiplication is just like ordinary multiplication. Let the child multiply the same quantity by 10, he gets Rs. 3289·50 which is same as Rs. 3289·5. Multiplying again by 10, he gets Rs. 32895 00 or Rs. 32895 only. Thus let the child observe that each multiplication by 10 just takes the decimal point one step to the right. Thus for multiplication by 100, the decimal point moves 2 steps to the right.

It is obvious that division by 10 would result in the decimal point moving one step towards the left and division by 100 would take it two points towards the left and so on.

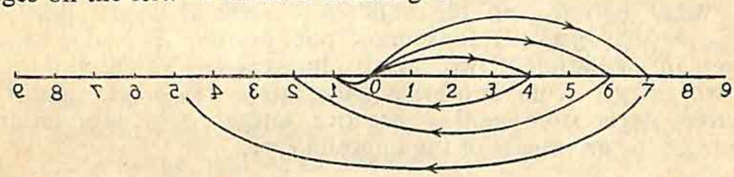
7

Integers

[EXPERIMENTS 61 – 70]

Experiment 61: Mirror numbers, subtraction with the help of mirror numbers

See 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,...... in a mirror and write their images on the left. 0 remains unchanged.



On this number line, we can subtract even greater numbers from smaller ones. Thus to get 4—5, we start from 0, move 4 steps towards the right and then 5 steps towards the left, we reach, so that

4 — 5 = ɪ

6 — 8 = ꙅ

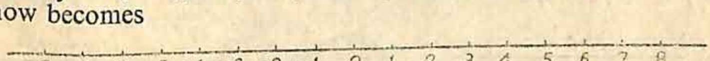
7 — 12 = ƨ

0 — 1 = ɪ

0 — 2 = ꙅ

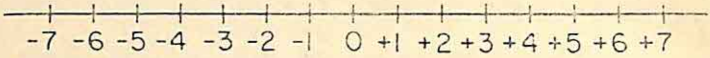
0 — 3 = Ɛ

We shall omit 0 and denote the images of 1, 2, 3, 4, ... on the left of 0 by -1, -2, -3, -4, ... respectively, so that our number line now becomes



More problems in subtraction can be done at this stage.

In many new books, the numbers on the left of zero are denoted by $-1, -2, -3, -4, \ldots\ldots$ and consistently with this, the numbers on the right of zero are denoted by $+1, +2, +3, +4, \ldots\ldots$, so that the number line becomes



These 'signed numbers' together with zero are called integers. The numbers with positive sign are called positive integers while the numbers with negative sign are called negative integers.

The children can do addition and subtraction problems like the following, on this number line

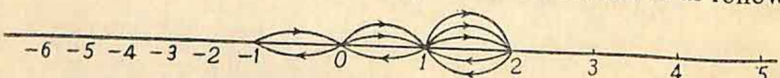
$$
\begin{aligned}
{}^{+}3 &+ {}^{+}4 = {}^{+}7 \\
{}^{+}3 &- {}^{+}4 = {}^{-}1 \\
{}^{+}4 &- {}^{+}3 = {}^{+}1 \\
{}^{+}3 &+ {}^{-}4 = {}^{-}1 \\
{}^{+}3 &- {}^{-}4 = {}^{+}7 \\
{}^{-}3 &+ {}^{-}4 = {}^{-}7 \\
{}^{-}3 &- {}^{-}4 = {}^{+}1
\end{aligned}
$$

What patterns do the children observe about (*i*) addition of two positive integers (*ii*) addition of one positive and one negative integers (*iii*) addition of two negative integers (*iv*) subtraction of one positive integer from another positive integer (*v*) subtraction of one negative integer from another negative integer (*vi*) subtraction of an integer for an integer of the opposite type.

The numbers on the left of zero can also be denoted by $\overleftarrow{1}, \overleftarrow{2}, \overleftarrow{3}, \overleftarrow{4}, \ldots$ and the numbers on the right of zero by $\overrightarrow{1}, \overrightarrow{2}, \overrightarrow{3}, \overrightarrow{4}, \ldots$ and similar addition and subtraction can be done with these 'arrowed' numbers or directed numbers. The positive integers can be called 'right' directed numbers and the negative integers can be called 'left' directed numbers. Children can make statements about addition and subtraction of these directed numbers. These can also interpret these as problems of motion towards the right or the left.

Experiment 62 : Moving on the number line with a coin

We can start from 0. We toss a coin. If it turns up heads (three lions), we move one step to the right ; if it turns up tails, we move one step to the left. We toss the coin again and move one step to the right or left of the point we had reached earlier according as it turns up heads or tails. This experiment is repeated fourteen or twenty times. Thus if we get heads, heads, tails, heads, tails, tails, tails, heads, heads, heads, the path of the counter is as follows,



and the final position is 2.

Each child does the experiment on his number line or comes to the board and does it. To find at any stage how many times the coin has been thrown, count the number of arrow-lines or the tally can be kept by drawing lines and then crossing them. Thus if a child has to throw the coin 10 times, he draws 10 lines and after 5 throws, his tally would look as follows :

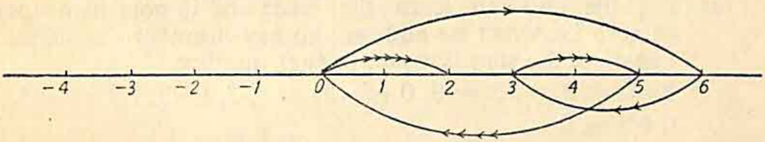
I *I* *I* *I* *I* 1 1 1 1 1

This means that he has to throw the coin 5 times more.

Experiment 63 : Moving on the number line

Suppose a drunkard starts from a point on a road, moves 6 steps towards the right, then 3 steps to left, then 2 steps to the right, then 5 steps to the left, then 2 steps to the right, where is he now ?

We represent his movement on the number line, starting with 0. It is obvious he is finally 2 steps towards the right of his original position,



6 steps to the right we denote by 6

3 steps to the left we denote by -3

2 steps to the right we denote by 2

5 steps to the left we denote by -5

2 steps to the right we denote by 2

and whole operation is denoted by $6-3+2-5+2$,

and since we are finally at 2, we say $6-3+2-5+2=2$

With the first step we write no sign if it is towards the right and $-$ sign if it is towards the left. With other steps we use $+$ sign if towards right, otherwise, we use $-$ sign.

Thus to find $-3-4$, we start with 0, move 3 steps towards the left, then another 4 steps towards the left to reach -7, so that

$$-3-4 = -7$$

Similarly
$$-5-6 = -11$$
$$5-6 = -1$$
$$-5+6 = 1$$

Let the children do some problems of both types.

(1) Problems in symbols, translated into problems of movement on the number line, and then solved.

(2) Problems of movement on the number line translated into problems in symbols and then solved.

Let each child find out after how many steps 'his' drunkard returns to the original position and let the children compare their results. The child whose drunkard comes last to the origin may be declared the winner. Different coloured counters may represent different drunkards.

In another game, each child throws a dice and a coin. If he gets '4' and a head, his counter moves 4 steps towards the right, if he gets '5' and a tail his counter moves 5 steps towards the left. The child whose counter is farthest from the origin after hundred steps is the winner. At each stage each child has also to speak out an addition or subtraction sum. Thus suppose he is at -3 and he gets 4 and a tail, he says $-3-4 = -7$ and if he gets 5 and a head, he says $-3+5 = 2$.

Experiment 64 : Addition table for integers

We explained the addition of positive and negative integers by using the number line in experiment 63.

(*a*) Let the children learn the basic and important property of zero *i.e.* when we add zero to any number, positive or negative, the sum is equal to that number. Thus
$3+0 = 3$, $0+3 = 3$, $0+(-3) = -3$, $(-3)+0 = -3$,
$0+0 = 0$

This can be illustrated by using the number line.

(*b*) Again by using the number line, let the children complete the following addition table :

+	−5	−4	−3	−2	−1		1	2	3	4	5
5	0	1	2	3	4	5	6	7	8	9	10
4	−1	0	1	2	3	4	5	6	7	8	9
3	−2	−1	0	1	2		4	5	6	7	8
2	−3	−2	−1	0	1	2	3	4	5	6	7
1	−4	−3	−2	−1	0	1	2	3	4	5	6
	−5	−4	−3	−2	−1	0	1	2	3	4	5
−1	−6	−5	−4	−3	−2	−1	0	1	2	3	4
−2	−7	−6	−5	−4	−3	−2	−1	0	1	2	3
−3	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2
−4	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
−5	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0

Let the children note that

(*i*) Each row gives the number line.

(*ii*) Each column gives the number line.

(*iii*) To every number, there is another number, which when added to first number gives zero :

What is the number which when added to 3 gives 0 ? In the row of 3, we look up for zero, we find it occurs in the column of -3. Therefore, -3 is the number which when added to 3 gives zero.

$$3+(-3) = 0.$$

Similarly what is the number which when added to -4 gives 0 ? We look up the row of -4. We find zero occurs in the column of 4. Therefore 4 is the required number. Zero is called the *additive identity* of the system of integers, because the sum is identical with the number added to zero. (-4) is called the additive inverse of 4 and 4 is called the additive inverse of -4.

(*iv*) Let the children note that

$$3+4 = 7$$
$$(-3)+(-4) = (-7)$$
$$(-3)+4 = 1$$
$$3+(-4) = -1$$

Let the children deduce the laws of addition.

(*a*) When we add two negative numbers, the sum is equal to the negative of (additive inverse of) the sum of the two corresponding positive numbers.

(*b*) When we add one positive and one negative number, the sum is equal to their difference in magnitude and the sign, is the same as that of the greater number.

(*v*) $$3+(-4) = -1$$
$$(-4)+3 = -1$$

Let the children verify that the commutative law holds for addition of integers.

(*vi*) $$[3+(-4)+(-5) = (-1)+(-5) = -6$$
$$3+[(-4)+(-5)] = 3+(-9) = -6$$

Let the children verify that the associative law holds for addition of integers.

Experiment 65 : The concept of order in natural numbers

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,......, are called **counting numbers** or **natural numbers**. If we consider the counting numbers, then the number which comes first in the counting process is said to be smaller than the number which comes afterwards in the process, and also the number which comes afterwards in this process is said to be greater than any number which comes earlier. Thus

We think	*We say*	*We write*
8 comes after 4,	8 is greater than 4,	$8 > 4$
10 comes after 6,	10 is greater than 6,	$10 > 6$
4 comes before 8,	4 is less than 8,	$4 < 8$
6 comes before 10,	6 is less than 10,	$6 < 10$

It may be noted that in the symbols $>$ and $<$ the vertex always points to the smaller number.

In the following examples, the children may fill in $>$, $<$, $=$ in the square

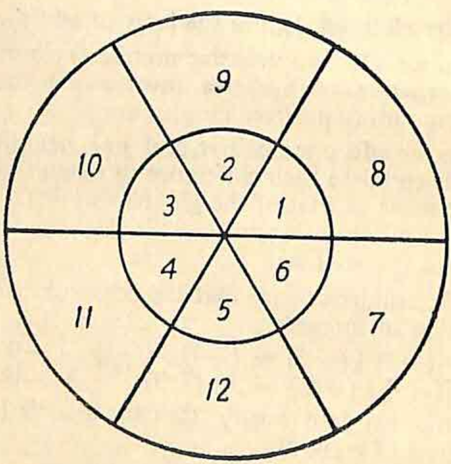
$3+4 \square \quad 5$
$4+10 \square \quad 16$
$5+5 \square \quad 10$

Construct a large number of such examples.

Experiment 66 : Making true sentences

Given numbers 1, 4, 8, 11 and the symbols $>$, $<$, $=$, what are the sentences one can make ?

$1+11 = 4+8$
$11-8 = 4-1$
$1+8 < 11$
$4+8 = 11+1$
$1+8 < 4+11$
$8+11 > 1+4$
and so on.



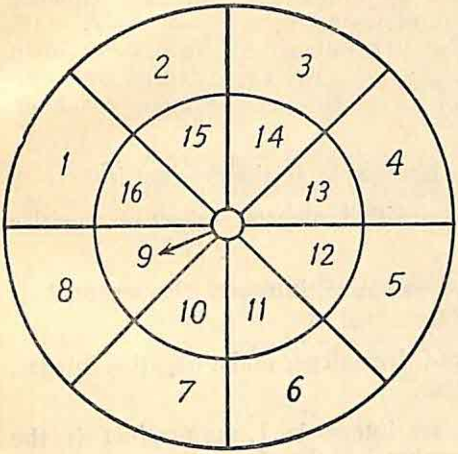
Ask the children to make as many sentences as possible, with each of the following :

(1, 4, 8, 11), (2, 5, 9, 12), (3, 6, 7, 10), and with the three symbols $>$, $=$, $<$.

A variation is the following :

Take 2 bags, one contains numbers say from 0 to 30 written on cardboards and the other contains only symbols $+$, $-$, $>$, $<$, $=$.

Each child chooses 3 numbers and 2 symbols and then makes as many different statements as he can with these by hanging them on the board.



Instead of cardboard or metal discs, one may use bottle tops painted with symbols and numbers.

Another variation is that a pointer may be attached at the centre. The child spins the pointer and connects those four numbers to which the pointer points when it stops.

The child may be encouraged to note that the sum of numbers on the left = the sum of numbers on the right in each case.

Experiment 67 : Order in integers

On the number line, any integer is said to be less than any of the integers which lie to its right and greater than all those which lie on its left. Thus

$-10 < -5,\ -5 < -3,\ -3 < -1,\ -1 < 2,\ 2 < 4, \ldots\ldots,$

$10 > 5,\ 5 > 3,\ 3 > 1,\ 1 > -1,\ -1 > -3$, etc.

Let the children give more examples. Let the children also notice that if first number is < a second number and the second number < a third number, then the first number < the third number and that a similar relation holds for >. This is known as *transitive law for the relation* < or > respectively. Does the law hold for the relation '=' ? Does the transitive law hold for human relations : 'Father of' 'mother of', 'brother of', 'cousin of', 'friend of' etc. ?

Experiment 68 : Multiplication of Integers

×	4	3	2	1	0	−1	−2	−3	−4
4	16	12	8	4	0	−4	−8	−12	−16
3	12	9	6	3	0	−3	−6	−9	−12
2	8	6	4	2	0	−2	−4	−6	−8
1	4	3	2	1	0	−1	−2	−3	−4
0	0	0	0	0	0	0	0	0	0
−1	−4	−3	−2	−1	0	1	2	3	4
−2	−8	−6	−4	−2	0	2	4	6	8
−3	−12	−9	−6	−3	0	3	6	9	12
−4	−16	−12	−8	−4	0	4	8	12	16

The children know the multiplication tables upto 10. They can complete easily the left hand corner square. Let them complete the first row. They note that they are moving on the number line towards the left of the number line by jumps of 4. In the sequence 16, 12, 8, 4, the succeeding numbers are 0, —4, —8, —12, —16. Similarly let them complete the first four rows. In moving down along the column of $-1, -2, \ldots$ etc., we move towards the right on the number line. The same applies to rows of negative numbers. Let the children note :

(*i*) When we multiply any number by 0, the product is 0.

(*ii*) When we multiply two positive integers, we get a positive integer.

(*iii*) When we multiply two negative integers, the product is again a positive integer.

(*iv*) When we multiply a positive integer and a negative integer, we get a negative integer.

(*v*) When we multiply a given integer by 1, the product is the given integer. The number 1 is therefore called *the multiplicative identity* of the number system.

(*vi*) Given an integer, does there exist an integer such that the product of the two integers is the multiplicative identity? No, except for the integers 1 and -1.

(*vii*) The commutative law holds for multiplication of integers.

(*viii*) $(-3)\times[4\times(-2)] = (-3)\times(-8) = 24$
$[(-3)\times 4]\times(-2) = (-12)\times(-2) = 24$

In every case the associative law can be verified.

(*ix*) $(-3)\times(-4+2) = -3\times(-2) = 6$
$(-3)\times(-4)+(-3)\times(2) = 12+(-6) = 6$

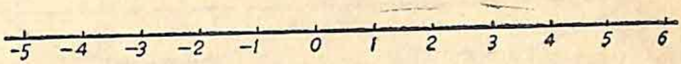
In every case the distributive law can be verified.

(*x*) Let the children complete the tables of multiplication from -10 to 10.

Experiment 69 : Illustrating laws of the system of integers geometrically with the help of the number line

We usually study arithmetic and geometry as two separate disciplines in elementary school mathematics. The two structures are however closely related and coordinated study of both is useful and illuminating. The present experiment is aimed at the children revising all the properties of the number system through the use of the number line. For each case, the children should give four or five illustrative examples,

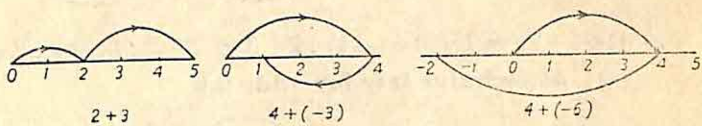
(*i*) The set of integers.................................... 3, −2, −1, 0, 1, 2, 3, 4, 5, ... can be represented on the number line.

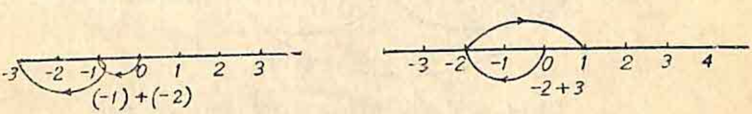


(*ii*) The positive integers are to the right of 0 and negative integers are to the left of 0.

(*iii*) An integer is greater than another integer if it is on the right of the second integer. An integer is less than any integer on its right and is greater than any integer on its left.

(*iv*) For addition of integers, we move towards the right for each positive integer, towards the left for each negative integer and do not move at all for 0. Thus :—



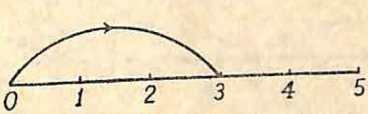


(*v*) Subtraction of an integer is defined as the addition of its additive inverse. The subtraction of 5 means addition of −5 and subtraction of −5 means addition of the additive inverse of −5 *i.e.*, 5. Thus :—

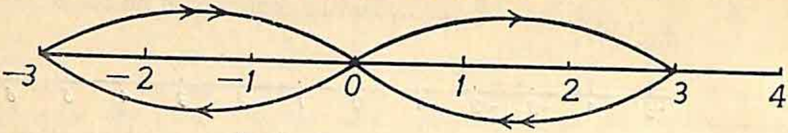
$$6-5 = 6+(-5), \quad 8-(-5) = 8+5$$

If for adding an integer, we move in one direction, for subtracting, we move in the opposite direction. For adding a positive integer, we move towards the right and therefore for subtracting a positive integer, we move towards the left. Similarly for adding a negative integer, we move towards the left and for subtracting a negative integer, we move towards the right.

(*vi*) Existence of zero *i.e.* $\square + 0 = \square$ for every integer *e.g.*



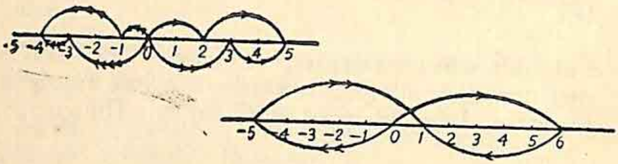
(*vii*) **Existence of an additive inverse**



$$3+(-3)=0$$
$$(-3)+3=0$$

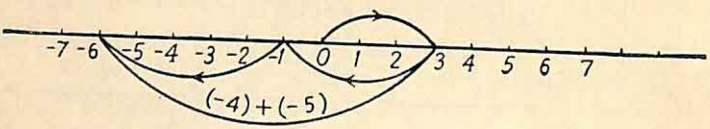
3 is the additive inverse of (-3) and (-3) is the additive inverse of 3.

(*viii*) **Commutative law for addition**



$(-1)+(-3) = (-3)+(-1)$; $2+3 = 3+2$; $6+(-5) = (-5)+6$

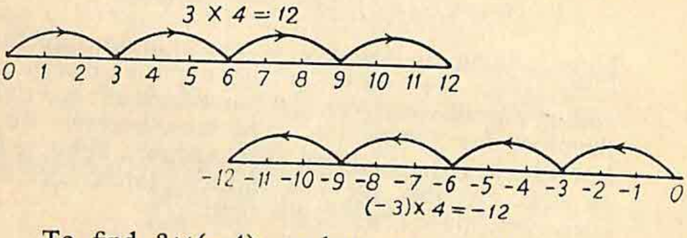
(*ix*) **Associative law for addition**



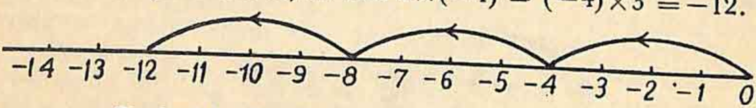
$$[3+(-4)]+(-5) = 3+[(-4)+(-5)].$$

(*x*) **Multiplication of Integers**

To find 3×4, we move 4 steps of length 3 each in positive (right) direction. To find $(-3)\times4$, we move 4 steps of length 3 each in the negative direction.



To find $3\times(-4)$, we have to use the commutative law (experiment 68) so that $3\times(-4) = (-4)\times3 = -12$.

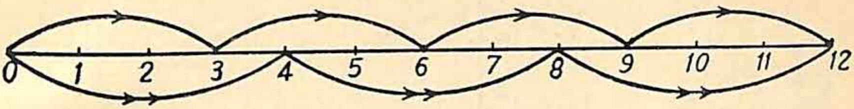


To find $(-3)\times(-4)$, we have again to use the result of experiment 68 to get $(-3)\times(-4) = 3\times4 = 12$.

(*xi*) **Commutative law for multiplication**

For positive integers, this is illustrated below

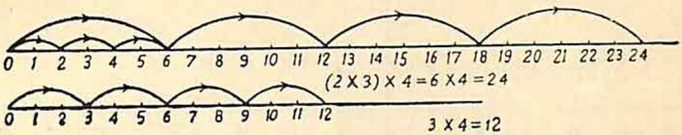
3×4

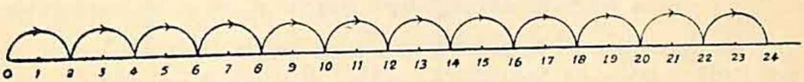


$4 \times 3.$

(*xii*) **Associative law for multiplication**

For positive integers, this is illustrated below



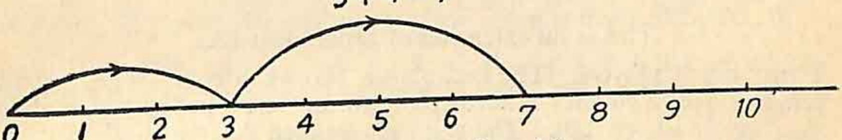


$$2 \times (3 \times 4) = 2 \times 12 = 24$$

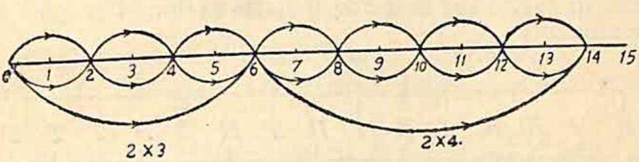
(*xiii*) **Distributive law for multiplication over addition**

For positive integers, this is illustrated below

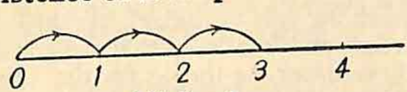
$3+4=7$



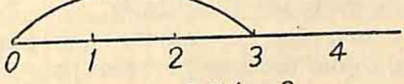
$$2 \times (3+4) = 2 \times 7 = 14$$



(*xiv*) **Existence of multiplicative identity**



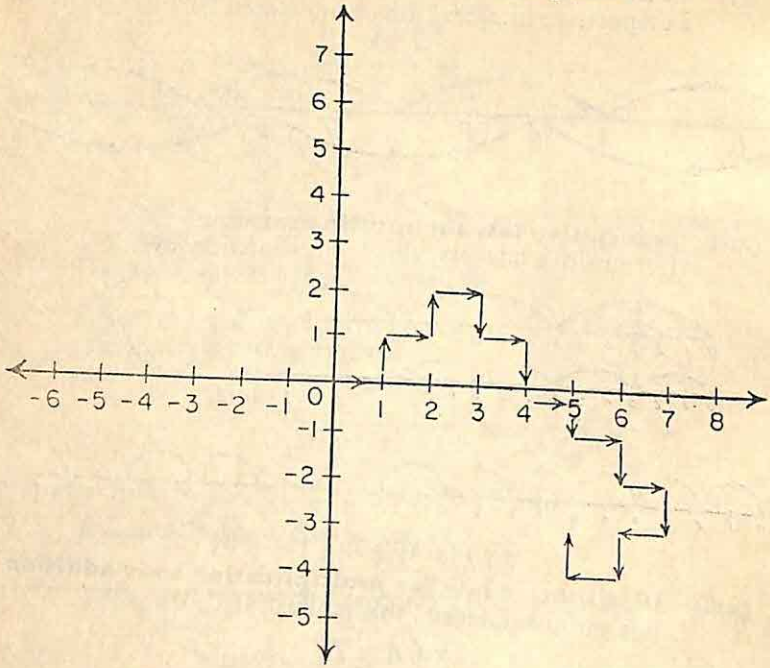
$1 \times 3 = 3$



$3 \times 1 = 3$

'1' is the multiplicative identity.

Experiment 70 : Drunkard problem on a plane



This is an extension of experiment 63.

There are 4 players. The first player throws a coin. The second player moves a counter towards the right or left one step according as it is head or tail. The third player then throws another coin. The fourth player moves the same counter up or down according as it is head or tail. The first player starts again. The path is shown for the sequence.

Third player

H H H H H T H T H T H T H T T T T H

First player

To check the final position, we consider the throws by the first player. There are 7 heads and 2 tails, so that the counter is 5 steps to the right. Similarly consider the throws for the third player. If there are more heads than tails, the net movement will be upwards, otherwise if there are less heads, the final position is below the horizontal number axis (as in the above case).

After 6 plays by each player, note the final position. Repeat this game five times and join the final positions to get a polygon.

If there are six players, the counter can also be moved above or below the plane.

8

Sets

[EXPERIMENTS 71—78]

Experiment 71 : Sets, union and intersection of sets

A set is a well defined collection of objects for which given any object, we can say whether it belongs to the set or not. Thus we have a set of students in a class, a set of benches, a set of chairs in a room, etc. The numbers 1, 2, 3 constitute a set which has 3 elements. The numbers 7, 8, 9, 10 constitute another set with 4 elements. If we combine these sets, we get another set with 7 elements *viz.*, 1, 2, 3, 7, 8, 9, 10. This set is called the *union* of the two sets. If A and B are two sets, their union is denoted by $A \cup B$.

The union of the sets 1, 2, 3, 4 and 3, 4, 5 is a set with 5 elements *viz.*, 1, 2, 3, 4, 5. Common elements are not repeated.

These two sets have 2 common elements. These 2 common elements constitute a set which is called the *intersection* of these sets. If A and B are two sets, their intersection is denoted by $A \cap B$.

The children should be encouraged to give examples of sets. The concepts of union and intersection of sets should be profusely illustrated.

What is the intersection set of the sets 1, 2, 3 and 7, 8, 9 ? These have no common element *i.e.*, their intersection set is a set with no elements. This set with no elements is called the *empty set* or the *null set* and is denoted by { } or ϕ.

Experiment 72 : Some special sets

We use the symbols 1, 2, 3, 4, 5,... etc. for counting numbers. These are called counting numbers or natural numbers. In this set, every number has a next number or a successor obtained by adding 1 to the preceding number. Thus after 99 we have $99+1=100$ and $100+1=101$, and this process of adding 1 can be continued indefinitely. The counting process thus can never come to an end.

This is expressed by saying that the set of natural numbers is infinite. This set is usually denoted by N.

The set obtained by taking the union of this set with the set consisting of the element 0 is called *the set of whole numbers*. It is the set consisting of 0, 1, 2, 3,... The infinite set consisting of -1, -2, -3, -4, ... is called *the set of negative integers*. The natural number set consisting of 1, 2, 3, 4, ... may also be called the set of positive integers. The set which is the union of the (*i*) set of the positive integers (*ii*) the set of negative integers (*iii*) set consisting of one element 0 is called the set of integers and is denoted by I.

The children may construct examples of successors and predecessors and may be given the concept of infinity as one involving a never terminating process. The teacher may discuss : Is the set of visible stars infinite ? Is the set of living human beings infinite ? etc.

Experiment 73 : Compositions in sets

Consider the set N of natural numbers. We can combine two of these numbers by the 'process' or 'operation' of addition. When we add two natural numbers, we get a natural number. This fact is expressed by saying that the set of natural numbers is '*closed*' under the composition of addition.

On the other hand, consider the operation of subtraction in the set of natural number. If we subtract a smaller number from a larger number, we get a natural number, but if we subtract a larger natural number from a smaller natural number we do not get a natural number. Thus the result of subtraction of two numbers of the set can be a number outside the set. Thus the set of natural numbers is not closed for the operation of subtraction.

Is the set 1, 2, 3, 4, 5 closed for addition ?

Is the same set closed for subtraction ?

Can a finite set be closed for addition ?

Is the set of integers closed for addition ?

Is it closed for subtraction ?

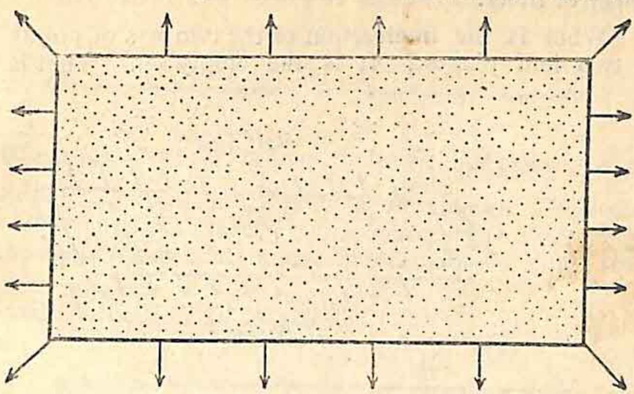
Is it closed for multiplication ?

Is it closed for division ?

Experiment 74 : Geometric objects as unions and intersections of sets of points

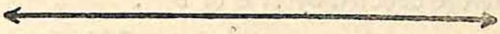
We do not define a point precisely. Points can however be represented by the tips of pencils, the ends of needles, the corners of a black board, or the corners where floor and walls meet. We represent a point by a dot. Let the children give more examples.

Ask the children to consider all points on the black board. These are infinite. Ask the children to consider the board to extend



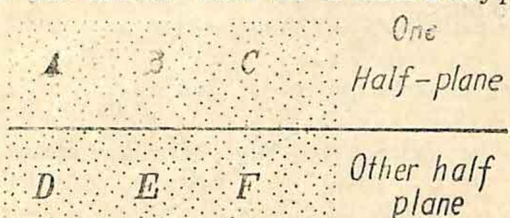
indefinitely upwards, downwards, towards right and left. The set of points obtained in this way is called a *plane*. Completely extended walls, floors, paper sheets, tops of tables are the examples of plane. Give other examples of planes. The children can give other examples of flat surfaces which are planes. The surface of a ball is not plane, it is a *curved surface*. To show that a plane is endless, you can draw arrows.

Take a paper sheet and fold it. The crease represents a *line*. A line is thus also a set of points. The line is said to lie in the plane. Since the plane extends to infinity, so does the line. We represent it as follows :



The arrows emphasize that the line goes to infinity in both directions. The children can give other examples of lines in the room.

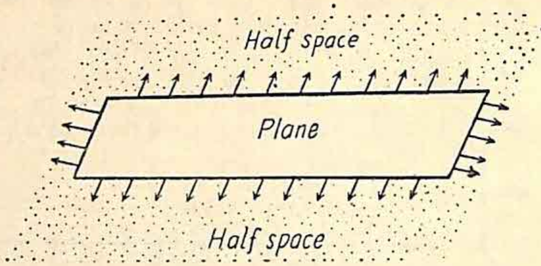
Draw a line on the board. This separates the set of points in the plane into two halves. Each set is called a *half plane*. Each



set contains an infinite number of points. Points A, B, C are in one half plane, D, E, F are in the other. An ant can go from a point in one half-plane to a point in the other half-plane only by crossing the line once or three or five or an odd number of times, while it can go from a point in one half-plane to a point in the same

half-plane either by not crossing the line or crossing it an even number of times.

What is the intersection of the two sets of points representing the two half planes ? It is the empty set. What is the union of



these sets ? Not the whole plane. The union of the two half planes and the line separating them gives the whole plane.

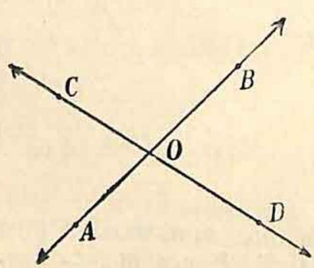
Similarly consider the set of points above the plane and below a plane. Each of them constitutes a half-space. Their intersection is null and the union of these half spaces and the plane is the whole space.

If we know two points on a line, we know the whole line. The

line is therefore denoted by $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$.

Experiment 75 : Intersection Sets of Lines and Planes

Consider the two sets of points in the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. What is the intersection of these two sets ? The intersection of these two sets is a single point O. This point is called the *point of intersection* of the two lines.

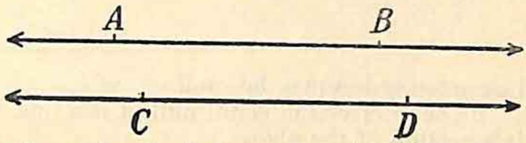


Consider a point O on the line AB. It separates the line into two *half lines* whose intersection is null and the union of the set of points in the two half lines and the singleton O is the whole line.

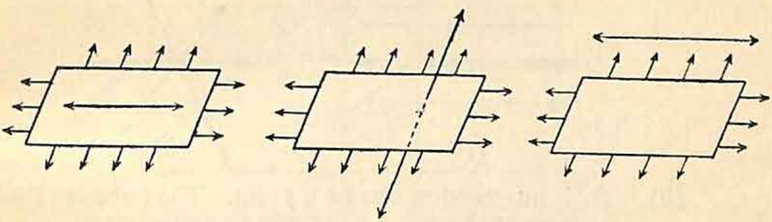
The union of O and one half line is called a ray and is represented as

and can be expressed as $\overrightarrow{OP}$, where P is any other point of the ray.

Consider two opposite edges of a foot rule. They are lines. What is their intersection ? Their intersection is null. Such lines as lie in the same plane and whose intersection is null are called *parallel lines.* $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.



Consider two edges in the room, one horizontal and one vertical, not meeting. They do not meet, yet they are not parallel, since they do not lie in the same plane. Thus two lines may or may not intersect. If they intersect their intersection is a single point and they lie in the same plane. If their intersection set is null, they may or may not be parallel. If they are parallel they lie in the same plane.



Consider the intersection set of a line and a plane. Ask the children the various possibilities, they can consider edges and walls in the room.

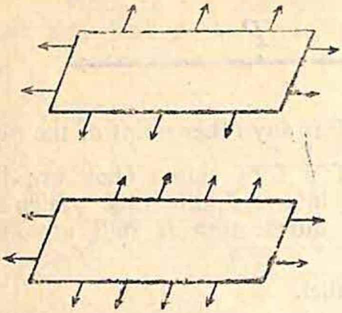
(*i*) If the intersection is null, the line is called parallel to the plane.

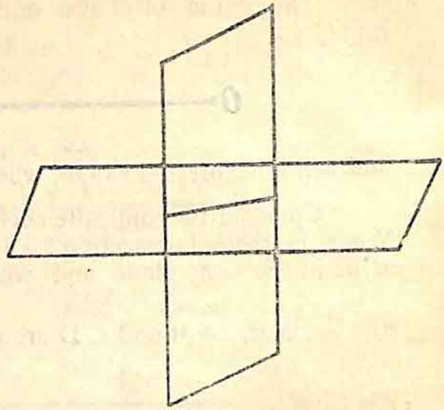
(*ii*) If the intersection is the whole line, the line is said to lie in the plane.

(*iii*) If the intersection is a single point, the point is called the point of intersection of the line and the plane.

Can a line and a plane intersect in two points only ? No, for if the two points A and B are on the plane, then the whole line AB is in the plane.

Next ask the children the possibilities of intersection, of two planes. Let them look at the walls.

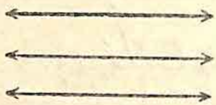




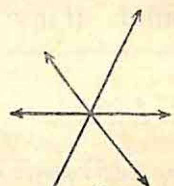
Their intersection may be null in which case the planes are parallel. If the intersection is not null, it is a line, which is called the line of intersection of the planes.

Consider three lines in a plane (on the black board). What can be their intersection ?

(*i*) The intersection of every pair can be null. The lines are then *parallel*.



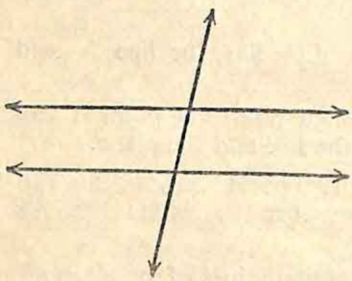
(*i*)

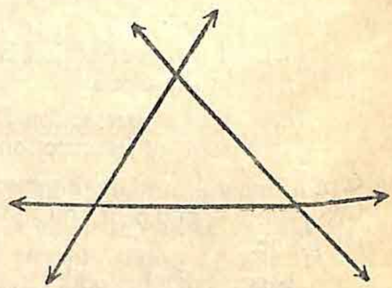


(*ii*)

(*ii*) Their intersection can be a point. The lines are then called *concurrent*.

(*iii*) The intersection of a pair can be null, but the two lines can have a non-null intersection with the third. The first pair is then parallel and then the third line is said to intersect them.



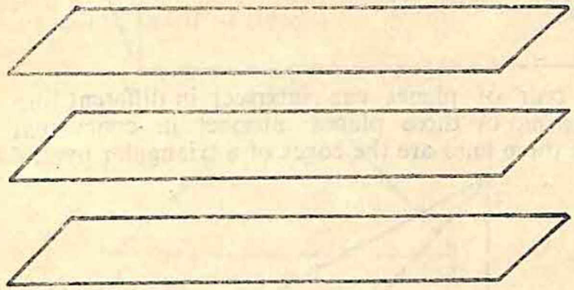


(*iv*) Every pair may have a non-null intersection.

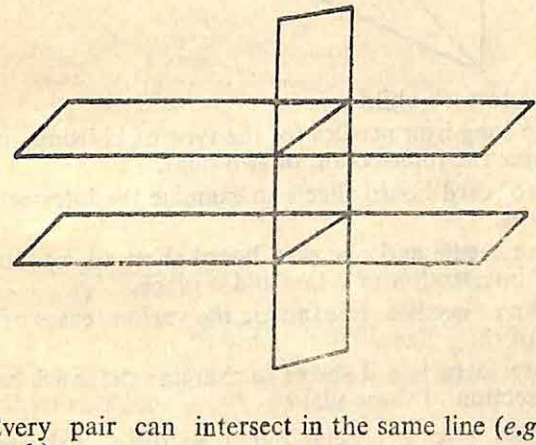
Next consider three planes. A pair of planes may intersect in a line or they are parallel.

Ask the children all possibilities about the intersection sets of these planes.

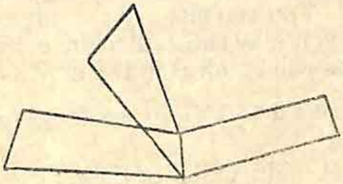
(*i*) The intersection of every pair can be null, in which case the three planes are parallel.



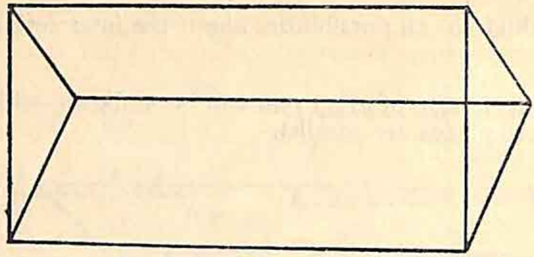
(*ii*) The intersection of one pair can be null and the third plane can intersect each of these two in a straight line.



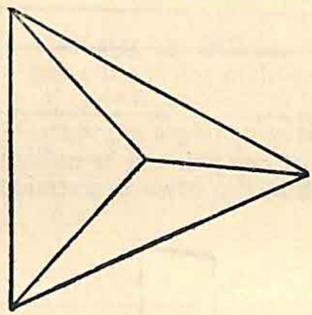
(*iii*) Every pair can intersect in the same line (*e.g.* sheets of a book).



(*iv*) Each pair can intersect in separate but parallel lines (giving rise to a triangular prism).



(*v*) Each pair of planes can intersect in different lines while each group of three planes intersect in concurrent lines. These three lines are the edges of a triangular pyramid.



You may give to each child :—

(*i*) First two long iron needles (of the type of knitting needles) to examine the intersection of two lines.

(*ii*) Then two card board sheets to examine the intersection of two planes.

(*iii*) Then one needle and one card board sheet to examine the cases of intersection of a line and a plane.

(*iv*) Then three needles to examine the various cases of intersection of three lines.

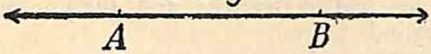
(*v*) Then three card board sheets to examine the various cases of intersection of three planes.

You may also give 2 sheets and 1 needle or 2 sheets and 2 needles or 3 sheets and 1 needle or 3 sheets and 2 needles and let the children discover all possibilities. You may be patient and let the children try and discover. You can also have ready-made apparatus for each possibility which you may show after the children have tried themselves. The visual aids will fix ideas in the minds of the children.

Experiment 76 : Polygons and Polygonal Regions

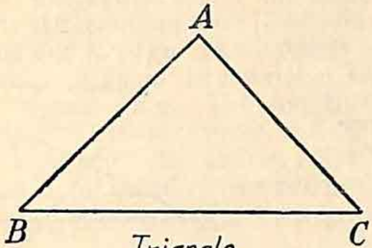
Consider the line $\overleftrightarrow{AB}$. The union of the set of points A, B and

all the points between A and B on the line is called the *line segment* $\overline{AB}$.



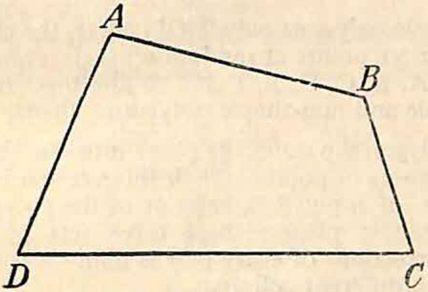
Line segment

Let A, B, C be three points in the plane, not lying in the same straight line. The union of the three line segments AB, BC and CA is called the *triangle ABC*. A, B, C are called vertices of the triangle and $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ are called its sides.



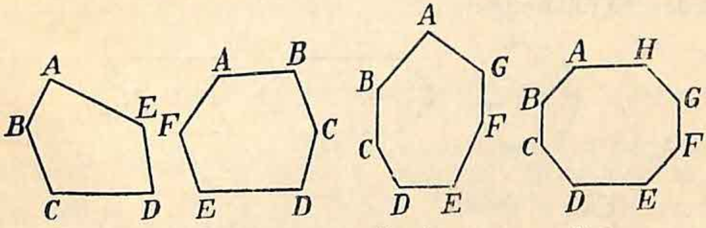
Triangle

Similarly a quadrilateral ABCD is the union of the line segments AB, BC, CD, DA where no three of the four points A, B, C, D are collinear.



Quadrilateral

Let the children define similarly a pentagon, a hexagon, a heptagon and an octagon.



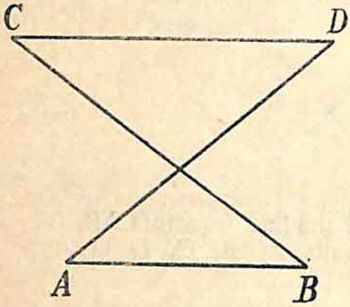
Pentagon Hexagon Heptagon Octagon

If we take any number of points A_1, A_2, A_3, A_4........, no three of which are collinear and join these in a cyclic order, joining the last point to the first, then union of all line segments gives a *polygon*. The points are called *vertices* and the line segments are called *sides* of the polygon.

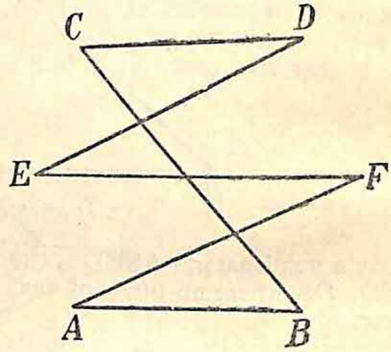
Thus Hexagon ABCDEF = AB $\cup$ BC $\cup$ CD $\cup$ DE $\cup$ EF $\cup$ FA.

A polygon in which no point except vertices occurs more than once is called a *simple polygon*. All the figures given on page 87 are simple. The following are not simple.

(iii)



(iv)

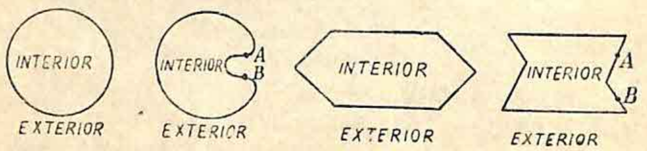


We consider simple polygons only. Of course, the children can be asked to plot five ar six points at random on their exercise books, name the points as A, B, C, D, E, F and to join these in order and let them discover simple and non-simple polygons.

Every simple polygon separates the plane into an interior and an exterior. Both are sets of points. Their intersection is null. The union of the interior of a polygon, exterior of the polygon and the polygon itself is the whole plane ; these three sets of points are disjoint *i.e.* the intersection of every pair is null. Let the children colour the three sets in different colours.

The union of a polygon and its interior is called a *polygonal region* (*triangular region, pentagonal region* or *hexagonal region* etc. as the case may be).

Consider the following :—



In drawing each of them :

(*i*) We can start from any point and end at the same point.

(*ii*) We have not to lift our pencil from the paper.

(*iii*) The pencil does not pass through any point twice, except the starting point.

Such curves are called *simple closed curves*. All simple polygons are simple closed curves, but the converse is not true.

Each simple closed curve separates the plane into an interior and an exterior. If we join a point in the interior to a point in the exterior, the line has to have a non-null intersection with the curve.

The union of a simple closed curve with its interior is called a *plane region*.

Though all the above curves are simple closed curves, the children may note some differences. If we join any two points on the boundaries of first and third curves, all points on this line segment except the points joined, lie in the interior. This does not happen in the second and fourth cases. The simple closed curves of the first and third type are called *convex curves*, and the corresponding regions are called *convex regions*. Of course, polygons which are simple convex closed curves are called *convex polygons*.

Let the children draw curves and polygons which are

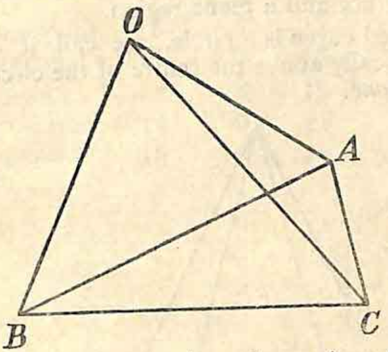
(*i*) open.
(*ii*) closed.
(*iii*) closed but not simple.
(*iv*) simple closed.
(*v*) simple, closed but not convex.
(*vi*) simple, closed and convex.

A circle is obviously a simple closed convex curve.

Experiment 77 : Polyhedra

(*a*) Pyramids and Cones

Draw a triangle ABC in a plane. Take a point O outside the plane. Join O to all points of the triangle.



Consider the union of the triangular regions OAB, OAC, OBC, ABC. This union is called a *triangular pyramid* ; O is called its *apex* and the triangular region ABC is called its *base*. O, A, B, C are called its *vertices*, segments OA, OB, OC, AB, AC, BC are called its *edges* and the four triangular regions are called its *faces*.

This triangular pyramid divides the whole space into three disjoint sets (*i*) the interior (*ii*) the pyramid (*iii*) the exterior. The union of the interior and the pyramid is called a *triangular pyramidal region.*

We may similarly draw any polygon *e.g.* a quadrilateral, a pentagon, a hexagon etc. in a plane and join the vertices of this to an apex O outside the plane to get a pyramid with a number of triangular faces and one polygonal face called the base.

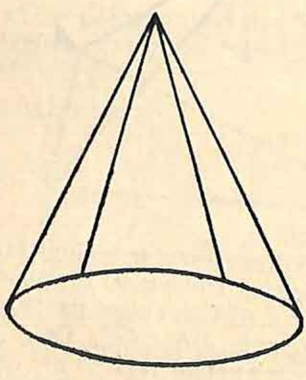
The teacher may show the following solids and ask the children to complete the table.

Pyramid	*Number of vertices* V	*Number of faces* F	*Number of edges* E	$V+F$	$V+F-E$
Triangular pyramid	4	4	6	8	2
Rectangular pyramid	5	5	8	10	2
Pentagonal pyramid	6	6	10	12	2
Hexagonal pyramid	7	7	12	14	2
Heptagonal pyramid	8	8	14	16	2
Octagonal pyramid	9	9	16	18	2

Let the children notice that in every case sum of number of vertices and faces exceeds the number of edges by 2. This is known as *Euler's Formula.*

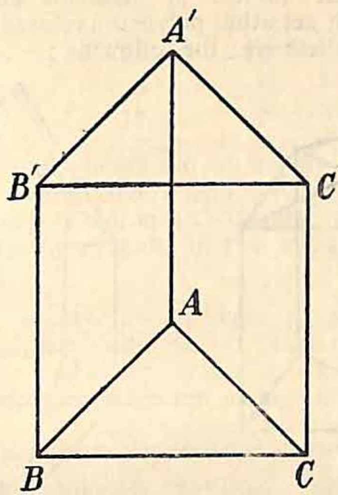
If we draw a simple closed curve in a plane and join every point of it to an apex outside the plane, we get a *cone* which is the union of the curved surface and a plane region.

If the simple closed curve is a circle, we call it *circular cone.* If the apex is just vertically above the centre of the circle, the cone is called a *right circular cone.*



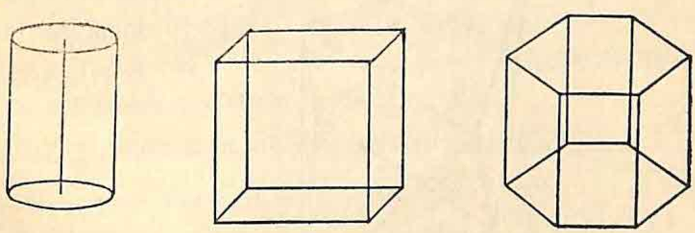
(*b*) **Prisms and Cylinders**

Take two planes, parallel or not and draw two triangles ABC and A′B′C′ on them. Join AA′, BB′, CC′, we get 2 triangular and 3 quadrilateral regions. Their union is called a *triangular* prism. Similarly we have quadrilateral prisms (including square prisms, rectangular prisms), pentagonal prism, hexagonal prism and so on.



Let the children complete the following table :—

Base	*Prism*	*V*	*F*	*E*	*V+F*	*V+F−E*
Triangle	Triangular	6	5	9	11	2
Quadrilateral	Quadrilateral	8	6	12	14	2
Square	Square	8	6	12	14	2
Rectangle	Rectangular	8	6	12	14	2
Pentagon	Pentagonal	10	7	15	17	2
Hexagon	Hexagonal	12	8	18	20	2
Heptagon	Heptagonal	14	9	21	23	2
Octagon	Octagonal	16	10	24	26	2



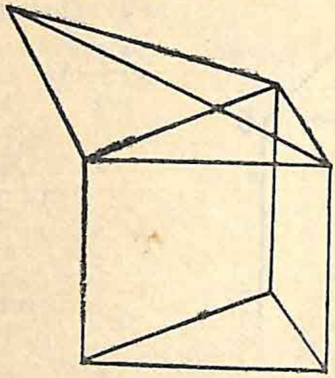
If the two bases are simple closed curves, then the union of the curved surface and the two base regions is called a *cylinder*.

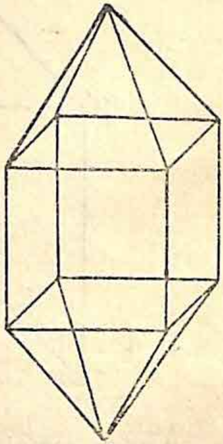
If the curves are circles, we get a *circular cylinder.*

If the lines joining the centres of the two equal circular bases is perpendicular to the planes of the bases, the cylinder is called a *right circular cylinder.*

(*c*) **Other Polyhedra**

We can have an infinite number of pyramids and an infinite number of prisms. We can get other polyhedra (closed surfaces with many faces) by combining these *e.g.*, the following :—



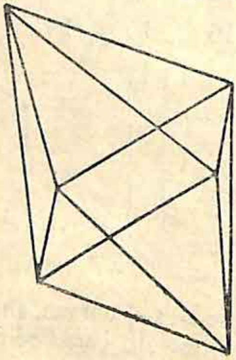


In each case, let the children verify that $V+F-E=2$.

(*d*) **Regular Polyhedra**

Can we have a polyhedron in which every face is the same regular polygon (3 equal sides or 4 equal sides or 5 equal sides) and for which the same number of edges meet at a vertex? We find that among the above polyhedra, only the triangular pyramid satisfies the condition. All its faces are triangular and at every vertex 3 edges meet.

Another one is the double square pyramid.



It has 8 triangular faces and at each vertex 4 edges meet. Since it has 8 faces it is known as *octahedron*.

There are only three other polyhedra satisfying the above conditions (what are these conditions ?). The five polyhedra are :—

Polyhedron	V	F	E	$V+F-E$
Tetrahedron	4	4	6	2
Cube	8	6	12	2
Dodecahedron	20	12	30	2
Octahedron	6	8	12	2
Icosahedron	12	20	30	2

These five polyhedra are known as regular polyhedra. These were known to the Greeks who were very much fascinated by these. The union of a polyhedron with its interior is called a regular solid. There are thus five regular or Platonic (after Plato, the great Greek thinker) solids.

Let the models of all these pyramids, prisms and regular polyhedra be shown to the class. These may be made of wood or of glass. A number of these are available as mineral shapes and therefore these models are made for geology laboratories.

The children can also make models of some of these by :—

(*i*) Pasting together card board pieces of the shapes of the faces. A systematic and simple method of construction will be given in Volume II.

(*ii*) Plasticine moulded to the shape of the polyhedral solids.

Experiment 78 : H.C.F. and L.C.M.

To find the highest common factor of a set of numbers, we find sets of factors of each number, find the intersection of these sets and then find the highest or largest number of this intersection set. Thus to find the H.C.F. of 8, 12, 16, 48, we have :

set of factors of 8 is {1, 2, 4, 8}

set of factors of 12 is {1, 2, 4, 6, 12}

set of factors of 16 is {1, 2, 4, 8, 16}

set of factors of 48 is {1, 2, 3, 4, 6, 8, 12, 16, 24, 48}

the intersection of these sets is {1, 2, 4}

The largest member of the intersection set is 4.

Therefore the H.C.F. of 8, 12, 16, 48 is 4.

Similarly to find the L.C.M. of a set of numbers, we find sets of multiples of the numbers, find their common set or intersection set

and then find the least member of the intersection set. Thus to find the L.C.M. of 8, 12, 16, 48, we have

the set of multiples of 8 is
{8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96,......}

the set of multiples of 12 is
{12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132,......}

the set of multiples of 16 is
{16, 32, 48, 64, 80, 96, 112, 128, 144, 160,......}

the set of multiples of 48 is {48, 96, 144, 192, 240, 288,......}

The intersection set of all these sets is {48, 96, 144, 192,... ..}

The least number of this intersection set is 48.

Therefore the L.C.M. of 8, 12, 16, 48 is 48.

Let the children do a number of examples of finding H.C.F. and L.C.M. from first principles.

Let the children also note here a set is written by enclosing its elements within curly brackets. Let them also note that ... stands for 'and so on'.

9

Truth Sets and Their Graphs

[EXPERIMENTS 79–88]

Experiment 79 : Making Sentences

Number	+ or −	Number	> or = or <	Number
(1)	(2)	(3)	(4)	(5)

The following types of activities are possible :

(*a*) Give (1), (2), (3), and (5) and ask the child to give (4) *e.g.*,

$7-4 \ \square \ 2$ (answer $>$)

$7+6 \ \square \ 13$ (answer $=$)

$7+5 \ \square \ 14$ (answer $<$)

(*b*) Give (1), (2), (3), (4) and ask the child to give (5) *e.g.*,

$7-4 = \square$ (answer 3)

$7-4 > \square$ (answer : 0 or 1 or 2)

$7-4 < \square$ (answer 4 or 5 or 6 or 7 or...)

(*c*) Given (1), (3), (4), (5), find (2) *e.g.*,

$10 \ \square \ 4 = 6$ (answer $-$)

$10 \ \square \ 5 = 15$ (answer $+$)

$10 \ \square \ 4 > 8$ (answer $+$)

$12 \ \square \ 8 < 5$ (answer $-$)

(*d*) Given (2), (3), (4), (5), ask the child to find (1) *e.g.*,

$\square + 5 = 6$ (answer 1)

$\square + 5 > 6$ (answer 2 or 3 or 4 or 5 or 6 or ...)

(*e*) Given (2), (4), (5), ask the child to find (1) and (3) *e.g.*,

$\square + \triangle = 5$ (answer 1 and 4, 4 and 1, 2 and 3, 3 and 2, 0 and 5, 5 and 0)

$\square + \triangle < 4$ {answer (0, 0), (0, 1), (1, 0), (1, 1), (1, 2), (2, 1)}

Again ask children to fill in the frames in the following

$6 \square 3 \square 2,$ $\square - 3 = 4,$ $3 \square 7 = 21$

$8 \square 2 \square 4,$ $\square \div 3 = 4,$

$9 \square 3 \square 2,$ $5 \square 4 = 20,$

$9 \div 3 \square 4,$ $20 \square 4 = 5,$

$$5 \square (6 \square 8) = (5 \square 6) + (5 \square 8)$$

Construct a large number of such examples.

Experiment 80 : Open Sentences and Truth Sets

Consider the sentences

$$3 + 2 = 4+1$$
$$20+ 5 > 20-5$$
$$3 + 10 < 10+3$$
$$5 + 6 > 3+9$$

The first and second are true sentences, the third and fourth are false. The children can construct a large number of examples of both *true and false sentences.*

Consider now the following sentences

$$\square > 5$$

It will be true if we write 6 or 7 or 8 or 9 etc. in the square. It will be false if we write 1 or 2 or 3 or 4 there. Thus this sentence is, as it is, neither true nor false.

It is an example of an *open sentence.* The set of numbers 6, 7, 8, 9, 10,......... for which it is true is called its truth set. We can have more examples.

Open Sentences	Truth Set
$\square < 5$	$\{0, 1, 2, 3, 4\}$
$\square \leq 5$	$\{0, 1, 2, 3, 4, 5\}$
$\square > 4$	$\{5, 6, 7, 8, 9...\}$
$\square \geqslant 3$	$\{3, 4, 5, 6,...\}$
$\square + 3 = 5+2$	$\{4\}$
$\square + \square = \square$	$\{0\}$
$\square + \triangle = 5$	$\left\{\begin{matrix}(0, 5), (1, 4), (2, 3), \\ (3, 2), (4, 1), (5, 0)\end{matrix}\right\}$
$\square + \triangle + \diamond = 5$	$\left\{\begin{matrix}(0, 1, 4), (1, 2, 2), (2, 3, 0), \\ ....\end{matrix}\right\}$
$\square = \square + 2$	$\{\ \}$
$\square - \square = -1$	$\{\ \}$
$\square + 3 = \triangle + 4$	$\{(1, 0), (2, 1), (3, 2),...\}$
$\square + 4 = 4 + \square$	the universal set *i.e.* set of all numbers
$\square + \triangle = \triangle + \square$	the universal set *i.e.*, set of all numbers.

$(\square+\triangle)+\diamond = \square+(\triangle+\diamond)$	the universal set *i.e.* set of all numbers
$(\square-\triangle)-\diamond = \square-(\triangle-\diamond)$	$\{(1, 2, 0), (2, 3, 0), (3, 3, 0), \ldots\ldots\}$
$\square-\triangle = 4$	$\{(5, 1), (6, 2), (7, 3), (8, 4), \ldots\}$

In each of the above open sentences, same number is to be written in each square ; similarly triangles have to have the same number and diamonds have to have same numbers. Numbers in squares, triangles and diamonds can be different.

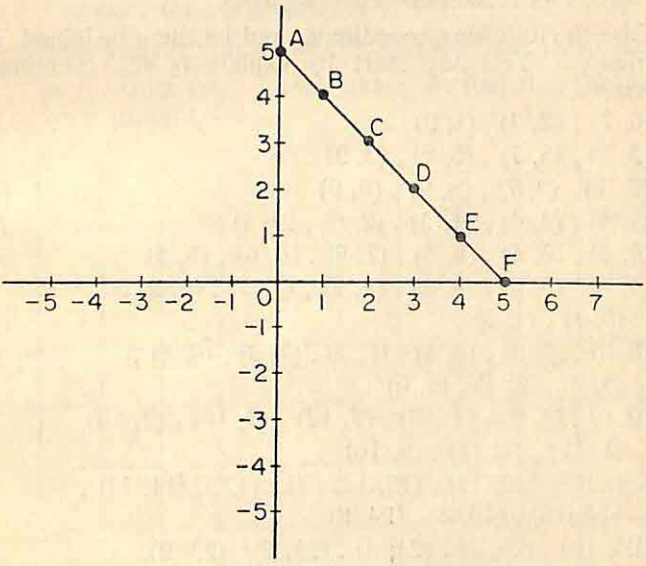
The children can construct more examples. Finding truth sets of open sentences is one of the most important activities in mathematics.

Experiment 81 : Joining lattice points

Consider $\square+\triangle=5$.

We considered this in experiments 79 and 80. We got the truth set : $\{(0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)\}$.

We can represent these on the number plane as follows :



Interpret (0, 5) as follows : Move 0 step towards the right and 5 steps upwards. We reach A. Similarly (1, 4) means : move 1 step towards the right and 4 steps upwards. We reach B. Similarly other solutions give C, D, E, F.

When we join them we find that the points lie on straight line.

The children may in the same way draw straight lines corresponding to $\square + \triangle = 6$, $\square - \triangle = 3$, $\square - \triangle = 4$ etc.

Experiment 82 : Truth sets of inequalities

Consider the open sentence

$$\square + \triangle < 5$$

What are the members of the truth set ? These are

(0,4)	(0,3)	(0,2)	(0,1)	(0,0)
(1,3)	(1,2)	(1,1)	(1,0)	
(2,2)	(2,1)	(2,0)		
(3,1)	(3,0)			
(4,0)				

Ask the children to show these points in the figure on page 97.

Ask whether any other point on the figure satisfies the above sentence. In fact all points below the line do it. What is the sentence that all points above the line satisfy ?

Obviously $\square + \triangle > 5$.

Similarly ask the children to graph the truth sets of

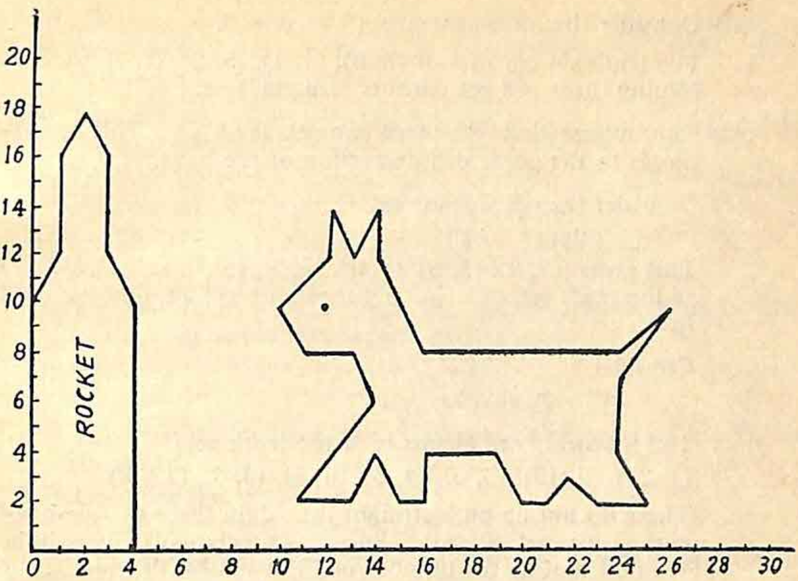
$$\square + \triangle > 6,\ \square + \triangle < 3,\ \square + \triangle \geqslant 5,\ \square + \triangle \leqslant 5.$$

Experiment 83 : Geometrical figures

Give the following coordinates and let these be joined in order to get figures. You may start by explaining the meaning of coordinates.

(0, 7) , (2, 7) , (1, 9)	*Triangle*
(3, 7) , (5, 7) , (5, 9) , (3, 9)	*Square*
(6, 7) , (9, 7) , (9, 9) , (6, 9)	*Rectangle*
(1, 4) , (3, 4) , (4, 5) , (2, 6) , (0, 5)	*Pentagon*
(6, 4) , (7, 4) , (8, 5) , (7, 6) , (6, 6) , (5, 5)	*Hexagon*
(1, 0) , (3, 0) , (4, 1) , (4, 2) , (2, 3) , (0, 2) , (0, 1) , (1, 0)	*Heptagon*
(6, 0) , (7, 0) , (8, 1) , (8, 2) , (7, 3) , (6, 3) , (5, 2) , (5, 1) , (6, 0)	*Octagon*
(0, 0) , (4, 0) , (4, 10) , (3, 12) , (3, 16) , (2, 18), (1, 16) , (1, 12) , (0, 10)	*Rocket*
(10, 10) , (12, 12) , (12, 14) , (13, 12) , (14, 14) , (14, 12) , (16, 8) , (24, 8) (26, 10) , (24, 7) , (24, 4) , (25, 2) , (23, 2), (22, 3) , (21, 2) , (20, 2) , (19, 4), (16, 4) , (16, 2) , (15, 2) , (14, 4) (13, 2) , (11, 2) , (14, 6) , (13, 8) , (11, 8) , (10, 10)	*Dog*

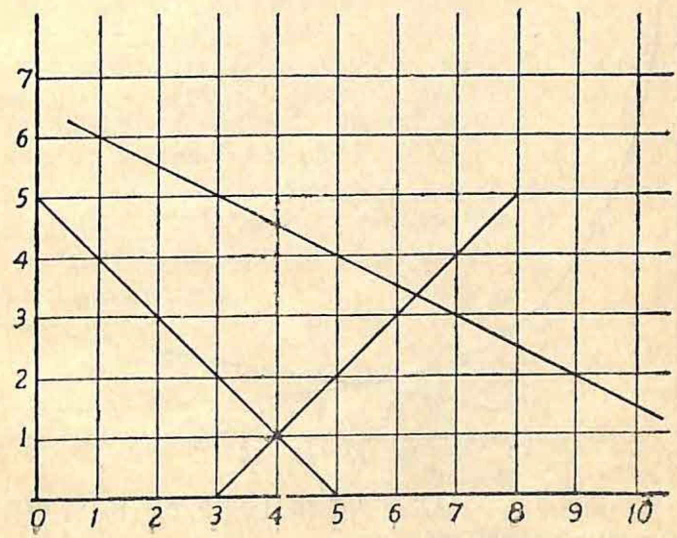
Make (12, 10) separately.



Experiment 84 : Drawing graphs of equations

(*i*) Consider the open sentence $\square + \triangle = 5$.

This gives {(0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)} as the truth set. Joining these, we find that the points lie on a straight line.



(*ii*) Consider the open sentence, $\square - \triangle = 3$.

The truth set consists of (3, 0), (4, 1), (5, 2), (7, 4), (8, 5),... Joining these, we get another straight line.

(*iii*) The intersection of these two sets is (4, 1). This corresponds to the point of intersection of the lines.

(*iv*) Consider the open sentence

$$\square + 2 \times \triangle = 13.$$

This gives {(1, 6), (3, 5), (5, 4), (7, 3), (9, 2), (11, 1), (13, 0)} as the truth set. On joining these, we get another straight line.

(*v*) Consider

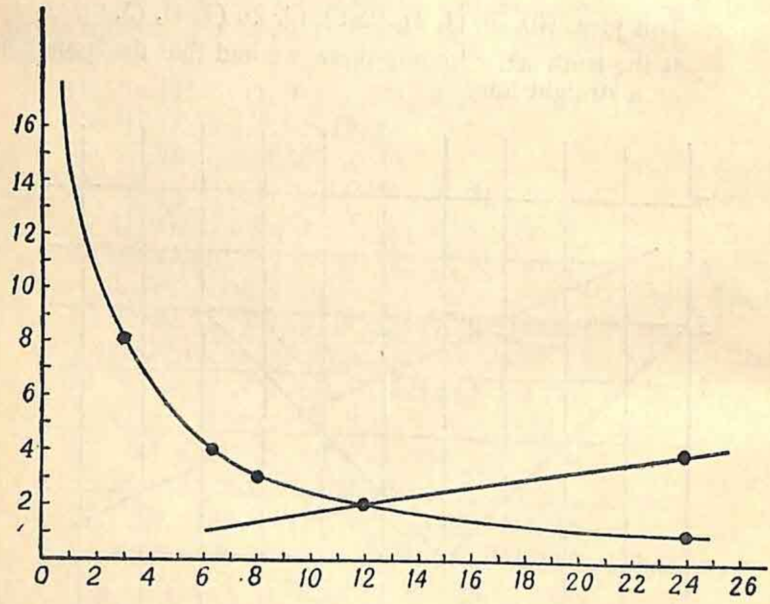
$$\square \times \triangle = 24.$$

The following are elements of the truth set:

(1, 24), (2, 12), (3, 8), (4, 6), (6, 4), (8, 3), (12, 2).

These do not lie on a straight line. Join these by free-hand curve, we get a curve known as rectangular hyperbola. We find that as the number in $\square$ increases, the number in $\triangle$ decreases.

(*vi*) Consider $\dfrac{\square}{\triangle} = 6$.



The truth set consists of (6, 1), (12, 2), (18, 3), (24,4),...... We get a straight line.

(*vii*) The set of points on the hyperbola of (*v*) and line of (*vi*) have a common point. The intersection of these sets is this point (12, 2).

(*viii*) The set of points on the lines $\square + \triangle = 5$ and $\square + \triangle = 6$ have no common element, their intersection is the null-set.

(*ix*) Let the children draw a large number of graphs and find the common elements of the sets of points on the graphs.

(*x*) Consider the graph of $\square + \triangle = 4$

and $2 \times \square + 2 \times \triangle = 8$

We find their graphs coincide. All the elements are common and belong to the intersection set.

(*xi*) Consider the graphs of $\square + \triangle = 13$

$\square \times \triangle = 36$

Their truth sets have two common elements *viz.* (4, 9), and (9, 4).

(*xii*) Consider the graphs of $\square = \triangle$

$2 \times \square = 2 \times \triangle$

Their truth sets have an infinite number of common elements *viz.* (1, 1), (2, 2), (3, 3) (4, 4),……

Thus the intersection sets of points in two graphs may have no element, or one element or more than one element or an infinite number of elements.

Experiment 85 : Drawing graphs of inequations.

Open sentences like

$\square > 5, \quad \square < 6$

$\square + \triangle > 7, \quad \square + \triangle < 8$

$\square + \triangle \leq 7, \quad \square + \triangle \leq 9$

are called inequations or inequalities. They are open sentences because for some numbers in frames, they may be true sentences, while for other numbers, they would be false.

Thus $8 > 5$ is true, $4 > 5$ is false.

The set of all these numbers or pairs of numbers for which the sentence is true is called the truth set of the sentence.

The following are examples of truth sets :

Open Sentence	Truth Set
$\square > 5$	{6, 7, 8, 9, 10,.......... }
$\square \geqslant 5$	{5, 6, 7, 8, 9,............ }
$\square + \triangle > 7$	{ (7, 1), (6, 2), (5, 3), (4, 4), (3, 5), (2, 6), (1, 7), (7, 2), (6, 3), (5, 4),........... }
$\square + \triangle \leq 7$	{ (7, 0), (6, 1), (5, 2),............ (6, 0), (5, 1),........... .... }

(*i*) The points corresponding to

$$\square + \triangle < 5$$

are shown by circles (0).

(*ii*) The points corresponding to

$$\square + \triangle > 5$$

are shown by crosses ($\times$).

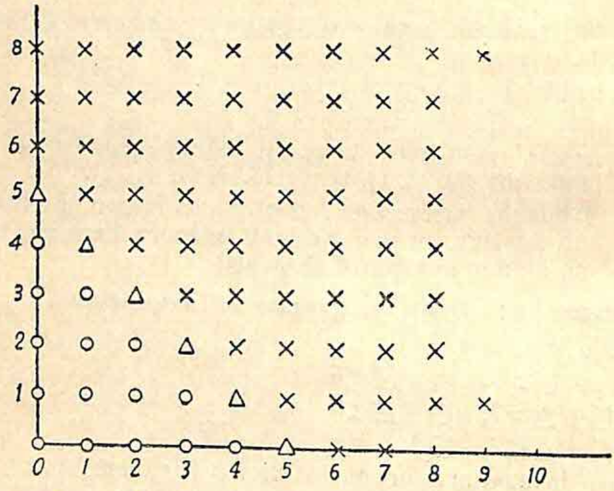
(*iii*) The points corresponding to

$$\square + \triangle = 5$$

are shown by triangles ($\triangle$).

(*iv*) The intersection of these three sets is null.

(*v*) The union of these sets is the universal set consisting of all points with non-negative integral coordinates.



(*vi*) The truth set of

$$\square + \triangle \leq 5$$

is given by circles and crosses together *i.e.* by the union of the truth sets of $\square + \triangle < 5$ and $\square + \triangle = 5$.

(*vii*) Similarly the truth set of

$$\square + \triangle \geqslant 5$$

is given by crosses and triangles *i.e.*, it is given by the union of solution sets of $\square + \triangle > 5$ and $\square + \triangle = 5$.

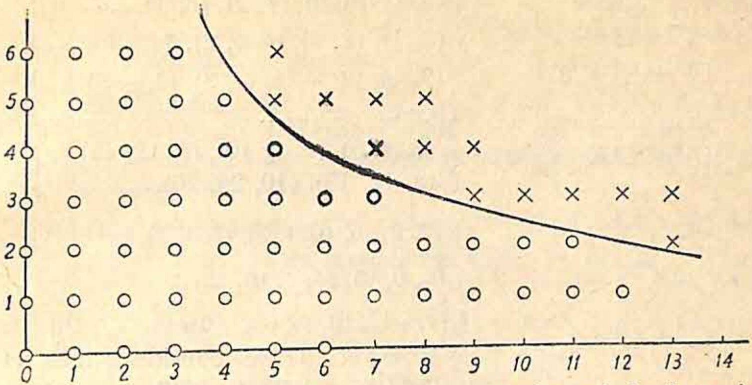
(*viii*) The truth set of

$$\square \times \triangle > 24$$

consists of crosses above the hyperbola, while the truth set of

$$\square \times \triangle < 24$$

consists of circles below the hyperbola.



(*ix*) It may be noted that we are considering only lattice points with non-negative integers as coordinates *i.e.* at present only such points can belong to our truth sets.

Experiment 86 : True sentences, false sentences, open sentences, further examples of truth sets of open sentences.

Write on the board sentences like

$5>3$, $5+9=6+8$, $7-3=8-5$, $3^2+4^2=5^2$, $2^2+3^2=4^2$,
$2^3+3^3=5^3$, $2^2=-4$, $21=1.3.7$, $28=1+2+4+7+14$.

and let the children state which sentences are true and which are false. Let them give more examples of true sentences and of false sentences.

Now write sentences such as

$\square > 5$. This will be true if in the frame we write, 6 or 7 or 8 or 9 etc. This will be false if in the frame we write, 1 or 2 or 3 or 4. Therefore this sentence can be true or false depending on what number we write in the frame. Such sentences are called open sentences, since their truth or falsehood is an open question. The set of values for which the sentence is true, is called its truth set. Give more examples like the following :

Open Sentence	Truth Set
$\square \geqslant 5$	$\{5, 6, 7, 8, 9, 10, \ldots\ldots\}$
$\square \times 3 = 24$	$\{8\}$
$\square^2 = -1$	ϕ (The square of an integer is never negative)
$\square < -1$	$\{-2, -3, -4, -5, \ldots\ldots\}$
$\square + \triangle = 5$	$\left\{\begin{matrix}(0, 5), (1, 4), (2, 3), (3, 2), (4, 1), \\ (5, 0), (6, -1), (7, -2), \ldots\ldots\end{matrix}\right\}$
$\square \times \triangle = 24$	$\left\{\begin{matrix}(4, 6), (6, 4), (8, 3), (3, 8), (12, 2), \\ (-3, -8)\ldots\ldots,\end{matrix}\right\}$

$\square + \triangle > 5$	$\{(5, 1), (6, 0), (7, 2), (8, 1), \ldots\ldots\}$
$\square + \triangle < 5$	$\{(3, 1), (2, 2), (-4, 2), (4, 0), \ldots\ldots\}$
$\square^2 + \triangle^2 = 25$	$\left\{\begin{matrix}(3, 4), (3, -4), (4, 3), (4, -3), \\ (-3, -4), \ldots\ldots\end{matrix}\right\}$
$\square^2 + \triangle^2 = \Diamond^2$	$\left\{\begin{matrix}(3, 4, 5), (6, 8, 10), (9, 12, 15), \\ (5, 12, 13), (10, 24, 26), \ldots\ldots\end{matrix}\right\}$
$\square + \triangle + \Diamond = 7$	$\{(7, 0, 0), (6, 1, 0), (5, 1, 1), (4, 2, 1)\}$
$\square + \triangle + \Diamond < 7$	$\{(6, 0, 0), (4, 1, 0), \ldots\ldots\}$
$\square^3 + \triangle^3 = \Diamond^3$	$\{(1, -1, 0), (2, -2, 0), (3, -3, 0), \ldots\ldots\}$ However if we consider the set of positive integers only, the truth set would be ϕ, since there are no three positive integers for which the sentence is true. This is known as Fermat's last theorem.
$\square^2 = \triangle$	$\{(1, 1), (3, 9), (5, 25), (-6, 36), \ldots\ldots\}$ (Note : square of an odd natural number is odd and square of an even natural number is even.)
$\square^2 = \square$	$\{(0, 0), (1, 1)\}$
$\left.\begin{matrix}\square \times \triangle = 36 \\ \square + \triangle = 13\end{matrix}\right\}$	$\{(4, 9), (9, 4)\}$
$\square^2 + \triangle^2 = 0$	$\{(0, 0)\}$
$\square^2 + \triangle^2 = -1$	ϕ or $\{\ \}$
$\left.\begin{matrix}\square + \triangle = 7 \\ 2 \times \square + 2 \times \triangle = 16\end{matrix}\right\}$	ϕ or $\{\ \}$
$\left.\begin{matrix}\square + \triangle = 7 \\ \square + 2 \times \triangle = 13\end{matrix}\right\}$	$\{(1, 6)\}$

Let the children note the following :

(*i*) When a sentence cannot be satisfied at all by natural numbers, we say its truth set is null in the domain of natural numbers.

(*ii*) An open sentence may not be satisfied by a natural number, but may be satisfied by negative integers. Its truth set is then not null in the domain of integers.

(*iii*) If an open sentence is not satisfied by any integer, its truth set is null in the domain of integers. The truth set may not be null in bigger domains which the child will study later.

(*iv*) When two sentences are given and we want to find their truth set, we can find the truth set of each sentence and find the intersection of these sets.

(*v*) To find truth set of $\square + \triangle \leqslant 5$, we can find the truth set of $\square + \triangle = 5$ and that of $\square + \triangle < 5$. The union of the truth sets of these two sentences gives the truth set of the original sentence.

(*vi*) The intersection of the truth set of $\square + \triangle \geqslant 5$ and $\square + \triangle \leqslant 5$ is the truth set of $\square + \triangle = 5$.

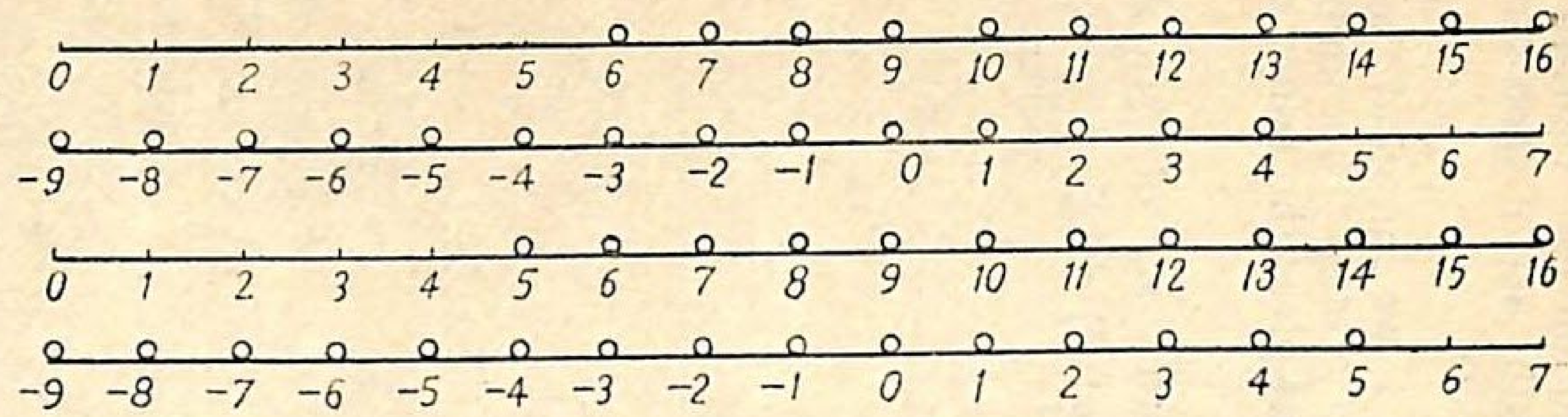
EXPERIMENT 87 ILLUSTRATING TRUTH SETS, UNIONS, INTERSECTIONS ON THE NUMBER LINE

Examples are the following

(a)

Open Sentence	Truth set
$\square > 5$	$\{6, 7, 8, 9, 10, \cdots\}$
$\square < 5$	$\{\cdots, -2, -1, 0, 1, 2, 3, 4\}$
$\square \geq 5$	$\{5, 6, 7, 8, 9, 10, \cdots\}$
$\square \leq 5$	$\{\cdots, -2, -1, 0, 1, 2, 3, 4, 5\}$

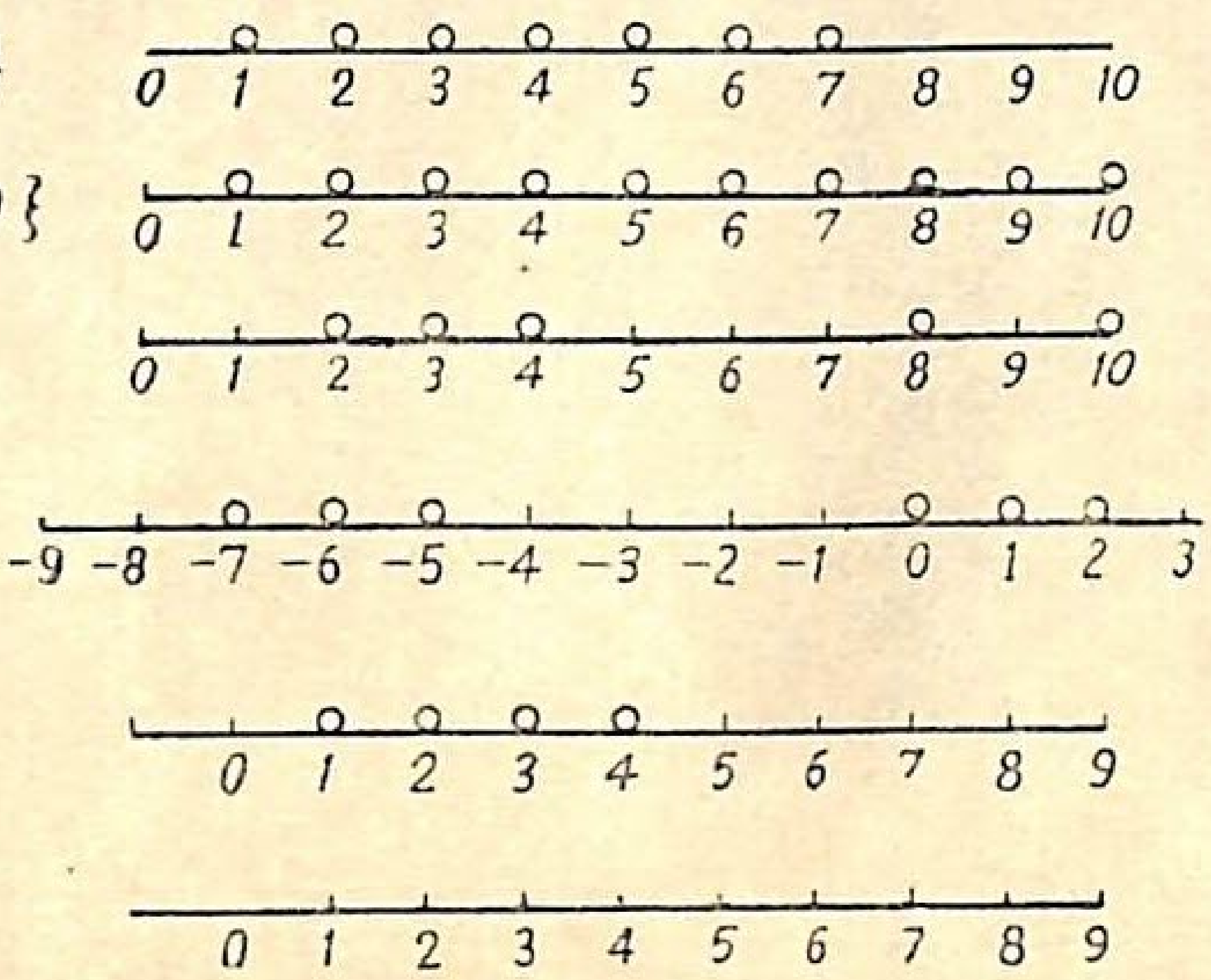
Geometrical Representation



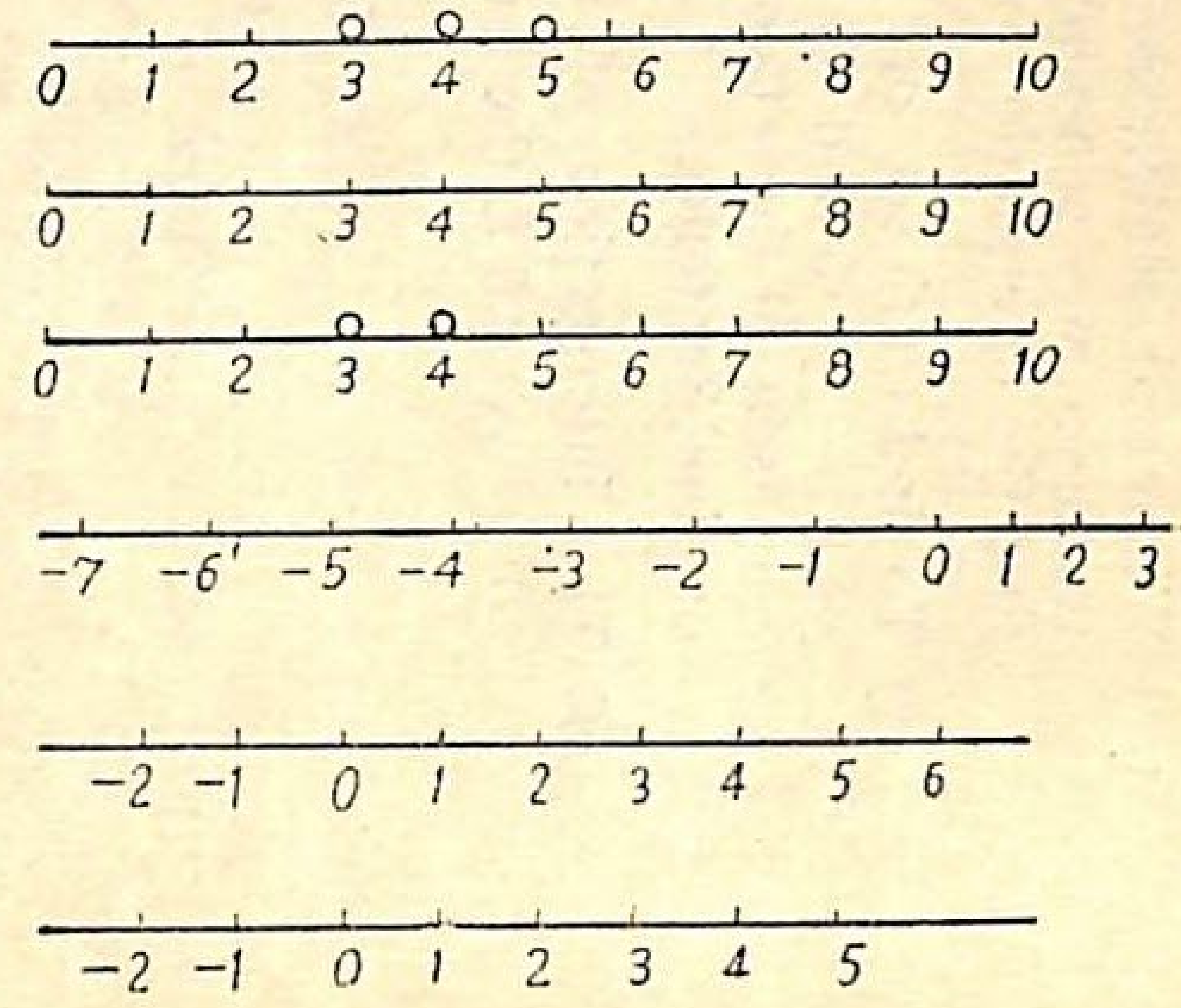
(b)

First set	Second set
$\{1, 2, 3, 4, 5\}$	$\{3, 4, 5, 6, 7\}$
$\{1, 3, 5, 7, 9\}$	$\{2, 4, 6, 8, 10\}$
$\{4, 10, 8, 3\}$	$\{2, 4, 3\}$
$\{-5, -7, -6\}$	$\{0, 1, 2\}$
$\{1, 2, 3, 4\}$	$\{\}$
$\{\}$	$\{\}$

Union

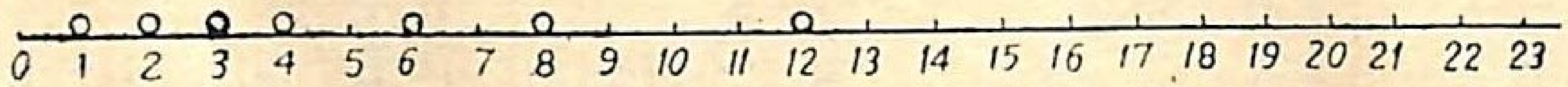


Intersection

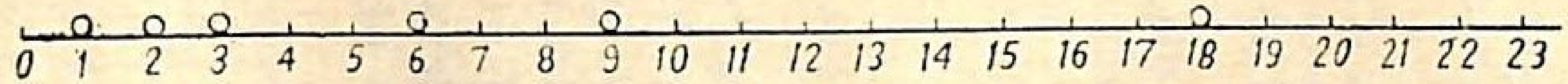


(c) H.C.F

Positive Factors of 24

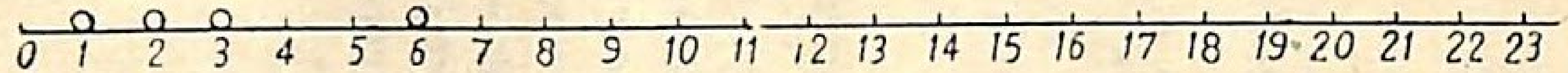


Positive Factors of 18



Positive Factors of 12

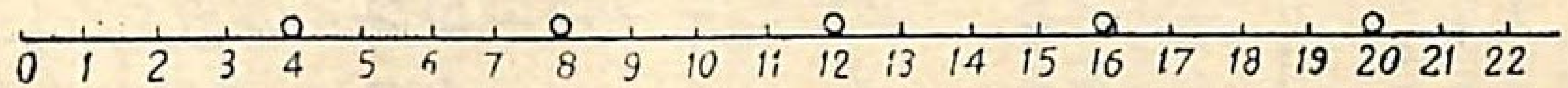
Intersection of these sets of factors



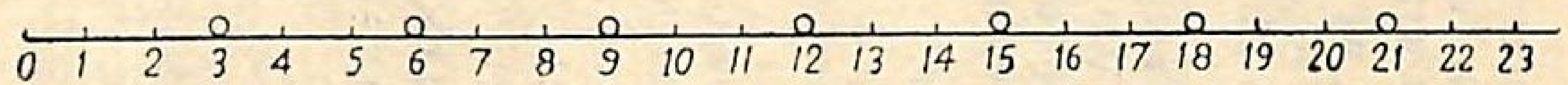
Highest common factor 6

(d) L.C.M.

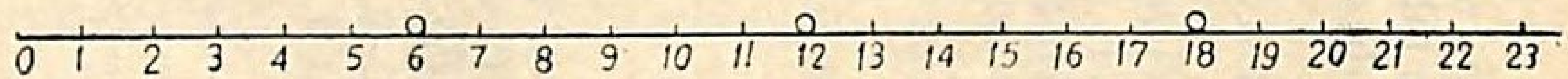
Positive multiples of 4



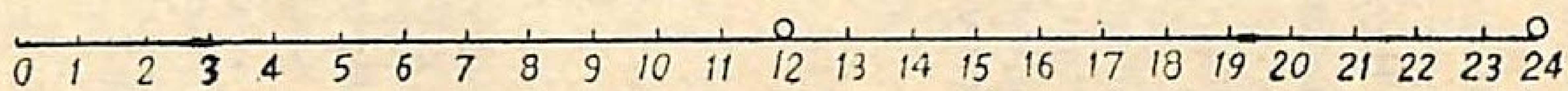
Positive multiples of 3



Positive multiples of 6



Intersection of these sets of multiples



Least common multiple 12

Experiment 88 : Use of letters of alphabet in open sentences

In our earlier discussions, we have used $\square$, $\triangle$, $\diamond$ etc. in open sentences. These were frames in which numbers could be written. For some numbers these gave true sentences, while for some others, they gave false sentences.

Instead of these, we can use a, b, c, d, ... or x, y, z...... etc. After earlier practice with $\square$, $\triangle$, $\diamond$ etc. children find no difficulty with the use of letters of the alphabet. $\square$, $\triangle$ etc. are place holders or position holders. We can write numbers in these frames. x, y, z are symbols which can be replaced by number from same number set which may be called replacement set. As examples, we have

Open Sentence	*Truth Set*
$x+2=5$	$\{3\}$
$x>5$	$\{6, 7, 8, 9, 10, \ldots\ldots\}$
$x+y=5$	$(x=1, y=4), (x=2, y=3), (x=3, y=2)$
$x+y=y+x$	Universal set (commutative law)
$x\times y=y\times x$	Universal set (commutative law)
$(x+y)+z = x+(y+z)$	Universal set (associative law)
$(x\times y)\times z = x\times(y\times z)$	Universal set (associative law)
$x\times(y+z) = (x\times y)+(x\times z)$	Universal set (distributive law)
$(x+y)\div z = (x\div z)+(y\div z)$	All integers except zero (distributive law)
$x^2\times x^3 = x^5$	Universal set
$x^3\times x^4 = x^7$	Universal set
$x = x+x$	$\{0\}$
$x = x+1$	$\{ \}$
$x\neq x$	$\{ \}$
$\left.\begin{array}{l} x\times y = 36 \\ x+y = 13 \end{array}\right\}$	$(x = 4, y = 9), (x = 9, y = 4)$.
$a\times(b+c+d)=(a\times b)+(a\times c)+(a\times d)$	Universal set
$(b+c+d)\times a=(b\times a)+(c\times a)+(d\times a)$	Universal set
$(a+b)\times(c+d)=(a\times c)+(a\times d)+(b\times c)+(b\times d)$	Universal set
$x^2+y^2 = z^2$	$(x = 3, y = 4, z = 5)$ $(x = 5, y = 12, z = 13)$ $(x = 6, y = 8, z = 10)$ ....

When the truth set of an open sentence is the universal set, we call it an identity. Which of the above sentences are identities ?

The teacher can construct a large number of such examples. The children can construct more themselves and give their truth sets.

10

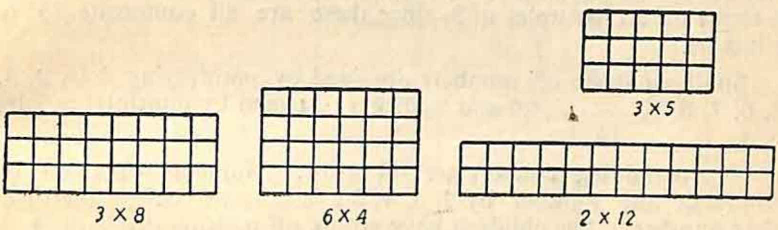
Some Number Theory

[EXPERIMENTS 89–92]

Experiment 89 : Prime and composite numbers.

Give the child 15 counters. He can arrange these in a rectangular array of 3×5. Similarly 24 counters can be arranged in

3×8 or 4×6 or 2×12 arrays



On the other hand, give the child 7 counters, he can arrange these only in 1×7 arrays. Similarly 11 or 13 or 17 counters can be arranged only in 1×11 or 1×13 or 1×17 arrays.

Numbers like 15 and 24 which can be expressed as product of two numbers both different from 1 are called *composite numbers*, while numbers like 11 or 13 or 17 which can be expressed as product of two numbers, one of which must be 1, are called *prime numbers*.

The number 1 itself is neither prime nor composite.

Ask the children to guess all prime numbers less than 100. Let the wrong conjectures by some children be corrected by other children.

Experiment 90 : The sieve of Eratosthenes

Write numbers 1 to 100 :

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Let one child strike off numbers obtained by multiplying 2 by 2, 3, 4, 5, 6, 7, 8,.................., 50, since these are all composite (2 is one factor). Again ask another child to strike off numbers obtained by multiplying 3 by 2, 3, 4, 5, 6, 7, 8, ..., 33 *i.e.* let him strike off all multiples of 3, since these are all composite (3 is one factor).

Similarly strike off numbers obtained by multiplying 5 by 2, 3, 4, 5, 6, 7, 8, 9, ..., 20 and numbers obtained by multiplying 7 by 2, 3, 4, ..., 14.

The remaining numbers are all prime. Numbers obtained by multiplying any number by 2, 3, 4, 5 , ... are called *multiples* of that number. The children have struck off multiples of 2, 3, 4, 5, 6, 7, 8, 9, 10. In fact when multiples of 2 are struck off, multiples of 4, 6, 8, 10 are automatically struck off. When multiples of 3 are struck off, multiples of 6 and 9 are also automatically struck off. We therefore ask the children to strike off the multiples of 2, 3, 5, and 7 only.

Check that

$2\times1+1$ is prime

$2\times3\times1+1$ is prime

$2\times3\times5+1$ is prime

$2\times3\times5\times7+1$ is prime

Give other similar prime numbers.

Experiment 91 : Fibonacci numbers

Consider the numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55,........;

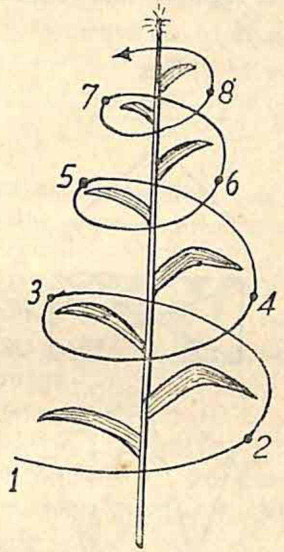
Encourage the children to recognize the patterns *e.g.*

(1) The numbers are increasing.

(2) Except the first and second, each number is the sum of the preceding two.

(3) If we take three numbers, square of the middle number is either one less or one more than the product of the other two numbers.

The numbers have botanical applications *e.g.* in ferns, in one full turn there are 2 leaves, in grasses, in one full turn there are 3 leaves, in beach in one full turn there are 3 leaves, in cherry in 2 full turns there are 5 leaves, in pears in 3 full turns there are 8 leaves and so on.



Encourage the children to form other number patterns *e.g.*

(*i*) Each number (except the first three may be the sum of the preceding three numbers).

1, 1, 1, 3, 5, 9, 17, 31, 57,.......................................

1, 2, 3, 6, 11, 20, 37, 68, ...

(*ii*) Each number is double of the preceding

1, 2, 4, 8, 16, 32, 64, ...

Similar patterns may be recognized in the following:

(*iii*) (1×1), (2×3), (3×5), (4×7), (5×9), (6×11), (7×13), (8×15),................................

The first number in the brackets increases by unity. The second number increases by 2 each time.

(*iv*) 1, 8, 27, 64, 125, 216,.....................................

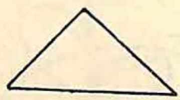
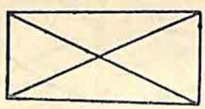
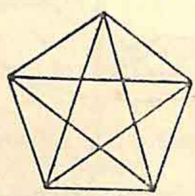
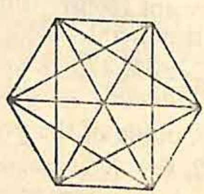
The numbers are cubes of 1, 2, 3, 4, 5, 6 ......respectively.

(*v*) 1, 3, 6, 10, 15, 21, 28, 36, 45, 55,..............................
Add successively 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

(*vi*) 1, 2, 6, 24, 120, 720,....................................
These are 1×1, 1×2, $1\times 2\times 3$, $1\times 2\times 3\times 4$, $1\times 2\times 3\times 4\times 5$,......................etc.

Give children a large number of such patterns, ask them to recognize these patterns and give the next three numbers in each case.

Give the children a specific number of points and let them join these by line segments and count the total number of line segments.

Number of Points	Lines	Number of Lines
2		1
3		3
4		6
5		10
6		15

Let them recognize that the pattern is the same as in (*v*) above.

Experiment 92. Tests of divisibility

In deciding whether a given number is prime or not and in some other problems, it is useful to be able to decide whether a given number is divisible by another number without actually dividing.

This can be decided by just considering the digits in the number. Thus we have the following rules :

(1) Every number is divisible by 1.

(2) Consider the number $d\,c\,b\,a$. This is equal to
$1000d+100c+10b+a$
$1000d$ is always divisible by 2.
$100c$ is always divisible by 2.
$10b$ is always divisible by 2.

$\therefore$ the given number would be divisible by 2 if and only if a is divisible by 2 *i.e.*, if $a=0$ or 2 or 4 or 6 or 8.

Thus a given number is divisible by 2 if the digit in the unit's place is 0 or 2 or 4 or 6 or 8.

(3) Consider again the number $d\,c\,b\,a$. This is equal to
$$1000d+100c+10b+a=(999d+99c+9b)+(d+c+b+a).$$
All the numbers in the first bracket are divisible by 3.
Therefore the given number would be divisible by 3 if $d+c+b+a$ is divisible by 3.

Thus a given number is divisible by 3 if the sum of its digits is divisible by 3.

(4) Consider again $1000d+100c+10b+a$
$1000d$ is always divisible by 4.
$100c$ is always divisible by 4.
Now $10b+a=8b+(2b+a)$
and $8\,b$ is always divisible by 4.
Therefore the given number is divisible by 4 if $2b+a$ is divisible by 4.

Thus a given number is divisible by 4 if the sum of its digits in the unit's place and twice the digit in the tenth place is divisible by 4.

(5) Again consider $1000d+100c+10b+a$
The first three numbers are always divisible by 5.
Therefore the given number would be divisible by 5 if a is divisible by 5 *i.e.* if $a=0$ or 5.

Thus a given number is divisible by 5 if the digit in the unit's place is 0 or 5.

(6) **A given number is divisible by 6 if it is divisible by both 2 and 3.** (Why ?)

(7) Now
$$100000f+10000e+1000d+100c+10b+a$$
$$=(100002f+10003e+1001d+98c+7b)$$
$$+(-2f-3e-d+2c+3b+a).$$

The numbers in the first brackets are all divisible by 7.

Thus the given number will be divisible by 7 if the number in the second bracket is divisible by 7. **Thus a given number is divisible by 7 if (digit in the unit's place) +3 (digit in the tenth place)+2 (digit in the hundredth place)−(digit in the thousandth place)−3 (digit in the ten thousandth place)−2 (digit in the hundred thousandth place) is divisible by 7.**

(8) Consider $1000d+100c+10b+a$

$=(100d+96c+8b)+(4c+2b+a)$

The numbers in the first bracket are all divisible by 8.

Therefore a given number is divisible by 8 if the sum (digit in the unit's place)+2 (digit in the tenth place)+4 (digit in the hundredth place) is divisible by 8.

(9) $1000d+100c+10b+a$

$=(999d+99c+9b)+(d+c+b+a)$

We easily deduce that a number is divisible by 9 if the sum of its digits is divisible by 9.

(10) **A number is divisible by 10 if the digit in the unit's place is 0.**

(11) $1000d+100c+10b+a$

$=(1001d+99c+11b)+(-d+c-b+a)$

A given number is divisible by 11 if the following sum is 0 or is divisible by 11 : (digit in the unit's place)−(digit in the ten's place)+(digit in the hundred's place)−(digit in the thousand's place)+( ) −( ).

We can similarly develop tests of divisibility by other numbers. Development of these tests gives the children a firm idea of the place value system.

(12) Since $7\times 11\times 13=1001$, a number which is divisible by 1001 is also divisible by 7, 11 and 13. In other words, a given number is divisible by 7, 11 or 13 if the number obtained by subtracting any multiple of 1001 from it is divisible by 7, 11 or 13.

Let the children

(*a*) apply each of these tests to a large number of examples; and

(*b*) note how these tests will determine the remainders in case the numbers are not divisible by a given number.

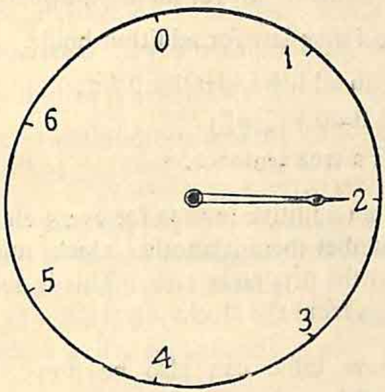
11

Modular Arithmetic and Logical Reasoning

[EXPERIMENTS 93—96]

Experiment 93 : Modular Arithmetic or Clock Arithmetic Modulo a Prime Number

Instead of the usual number line, we consider a round number line or a number clock with only a finite number of numbers.



Consider the clock with only 7 numbers 0, 1, 2, 3, 4, 5, 6. To add two of these numbers, we move the pointer from 0 as many steps as the first number and then move it further as many steps as the second number. Thus to find $3+6$, we move the pointer first 3 steps and then 6 further steps. The pointer reaches 2 so that in our clock $3+6=2$. In this way, let the children complete the table given on the next page :

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Let the children notice the following :—

(*i*) The system is closed for addition, *i.e.*, if we add two numbers of the clock, we get a number of the clock.

(*ii*) The commutative law for addition holds.

(*iii*) The associative law for addition holds.

(*iv*) There is an additive identity 0 *i.e* ,

$$\square+0=0+\square=\square$$

is always a true sentence.

(*v*) There is an additive inverse for every element *i.e.*, for every clock number there is another clock number which when added to the first gives zero. This is seen from the table as well as from the clock.

(*vi*) The above table can also be prepared by using the rule that to add any two numbers of the clock, we add them in the usual way and then subtract 7 or a multiple of 7. This type of addition is called addition modulo 7.

Next ask a child to multiply 3 by 4 on this clock *i.e.*, let the pointer move 4 steps of 3 each. It goes to 5, so that $3\times4=5$. Let the children complete the multiplication table given on the next page:

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Let the children notice that

(*i*) The system is closed for multiplication *i.e.*, when we multiply two numbers of the clock, we get a number of the clock.

(*ii*) The commutative law for multiplication holds.

(*iii*) The associative law for multiplication holds.

(*iv*) The distributive law holds for multiplication over addition.

(*v*) There is a multiplicative identity *i.e.* a number 1 such that

$$\square \times 1 = 1 \times \square = \square$$

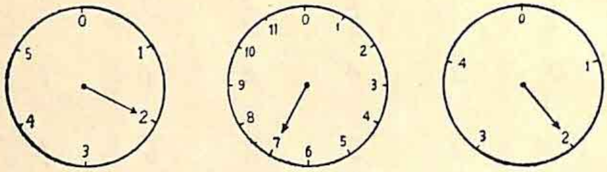
(*vi*) For every element except '0', there is a multiplicative inverse *i.e.* given any number except 0, we can find another number such that the product of the two is '1'. Thus $2 \times 4 = 1$, $3 \times 5 = 1$, $6 \times 6 = 1$. Thus 2 and 4 are inverses of each other, 3 and 5 are inverses of each other, 6 is inverse of itself.

(*vii*) Multiplication of two numbers is modulo 7 *i.e.* we can multiply two numbers in the ordinary way and subtract a multiple of 7, so that the remainder is one of the clock numbers. Thus 2×5 would be 10.

If we subtract 7, we get 3 so that $2 \times 5 = 3$.

Experiment 94 : Modular Arithmetic : Modulo a Composite Number

The children can construct other modulo clocks, like the following :



Modulo 6 clock   Modulo 12 clock   Modulo 5 clock

For each of these clocks, let the children complete addition and multiplication tables and let them notice that the above properties, except one, always hold. The exception is (*vi*) for multiplication. Thus for modulo 6, we find

$$2\times1=2,\ 2\times2=4,\ 2\times3=0,\ 2\times4=2,\ 2\times5=4,$$

we never get 1 as the product. Multiplicative inverse does not exist for any number except 1.

We also notice that in modulo 6 clock

$$2\times3=0$$

i.e., though neither factor is zero, yet the product is zero. This, of course, cannot happen in the ordinary arithmetic of the number line.

(*Contd.*)

Experiment 95 : Experiments in Logical Reasoning

The above experiments involve a good deal of logical reasoning. Logical reasoning is the heart of mathematics. Children should be given a great deal of practice in it.

Logical reasoning consists in deducing further facts from given arithmetic facts. We give below some examples. The teacher can construct a large number of similar examples.

Given	*We Deduce*	*Reasoning Used*
$33+66 = 99$	$66+33 = 99$	Commutative law for addition
$33\times 60 = 1980$	$60\times 33 = 1980$	Commutative law for multiplication
$(30+40)+50 = 120$	$30+(40+50) = 120$	Associative law for addition
$(3\times 4)\times 5 = 60$	$3\times(4\times 5) = 60$	Associative law for multiplication
$(3\times 4)+(3\times 5) = 27$	$3\times 9 = 27$	Distributive law
$3\times 9 = 27$	$(3\times 3)+(3\times 6) = 27$	Distributive law
$3\times 50-3\times 40 = 30$	$3\times 10 = 30$	Distributive law
$65\div 5 = 13$	$50\div 5+15\div 5 = 13$	Distributive law
$6+8 = 14$	$6+9 = 15$	Adding 1 to both sides
$99+98 = 197$	$100+100 = 200$	Adding 1+2 to both sides
$100+1000 = 1100$	$99+999 = 1098$	Subtracting 1+1 from both sides
$6+8 = 14$	$60+80 = 140$	Multiplying both sides by 10.
$50+30 = 80$	$10+6 = 16$	Dividing both sides by 5
$5+9 = 14$	$14-9 = 5$	Subtracting 9 from both sides
$20-13 = 7$	$7+13 = 20$	Adding 13 to both sides
$6+6+6+6+6 = 30$	$6\times 5 = 30$	Repeated addition is multiplication
$20-5-5-5-5 = 0$	$20\div 5 = 4$	Repeated subtraction gives division
$3\times 4=12$ and $5\times 6= 30$	$3\times 4\times 5\times 6=12\times 30=360$	Product of left hand sides=product of right hand sides
$8\times 7 = 56$	$56\div 7 = 8$	Definition of division
$7\times 9 = 63$	$63\div 9 = 7$	Definition of division
$56\div 7 = 8$	$7\times 8 = 56$	Definition of division

Given		We Deduce	Reasoning Used
$30>25$		$4\times 30>4\times 25$	Multiply both sides of the inequality
	or	$30+30+30+30>25+25+25+25$	by the same positive integer
$7<9$		$7+7+7<9+9+9$	Multiply both sides of the inequality
	or	$7\times 3<9\times 3$	by the same positive integer
$30>25$		$30-5>25-5$	Subtract same number from both sides
$120<130$		$120\div 5<130\div 5$	Dividing both sides by same positive
	or	$24<26$	integer
$20<30$		$25<35$	Adding same number to both sides
$1+2+3+4+5=15$		$10+20+30+40+50=150$	Multiplying both sides by 10
$1+2+3+4+5=15$		$11+12+13+14+15=65$	Adding $10+10+10+10+10$ on one side and 50 on the other
$1+2+3+4+5=15$		$21+22+23+24+25=115$	Adding $20+20+20+20+20$ on one side and 100 on the other
$1+2+3+4+5=15$		$2+3+4+5+6=20$	Subtracting 1 and adding 6
$625>500$		$625\div 5>500\div 5$	Dividing both sides by the same
	or	$625\div 5>100$	positive integer
$625<700$		$625\div 5<700\div 5$	Dividing both sides by the same
	or	$625\div 5<140$	positive integer
$-4<-3$		$4>3$	Multiplying by -1 changes sign of
$5>4$		$-5<-4$	inequality.

Experiment 96 : Implication.

If from a certain statement P, another statement Q can be deduced, we write P $\Rightarrow$ Q and read it as 'P implies Q'.

If from the statement Q, we can deduce statement P, we of course write Q $\Rightarrow$ P.

If Q can be deduced from P and also P can be deduced from Q we write P $\Leftrightarrow$ Q and read it as 'P implies and is implied by Q'.

Give the children the following and other similar examples:

(i) $x=5 \Rightarrow 2x=10$

$2x=10 \Rightarrow x=5$

$x=5 \Leftrightarrow 2x=10$

$2x=10 \Leftrightarrow x=5$

(ii) $x=5 \Rightarrow x^2=25$

$x^2=25 \not\Rightarrow x=5$

$x^2=25 \Rightarrow x=5$ or $x=-5$

(iii) $x>y \Leftrightarrow x+z>y+z$

$x>y \Leftrightarrow xz>yz$ if $z>0$

$x>y \Leftrightarrow xz<yz$ if $z<0$

$x>y \Leftrightarrow x-z>y-z$

(iv) $x>y \Leftrightarrow -x<-y$

$x<y \Leftrightarrow -x>-y$

(v) $x>y$ and $y>z \Rightarrow x>z$

(vi) $x\times 0=y\times 0 \not\Rightarrow x=y$

(vii) $x=3,\ y=4,\ z=5 \Rightarrow x^2+y^2=z^2$

$x^2+y^2=z^2 \not\Rightarrow x=3,\ y=4,\ z=5$

(viii) $x=4,\ y=2 \Rightarrow x+y>5$

$x+y>5 \not\Rightarrow x=4,\ y=2$

(ix) Two lines are parallel $\Rightarrow$ their intersection set is a null set. Intersection set of two lines is a null set $\not\Rightarrow$ the lines are parallel.

(x) It is raining heavily $\Rightarrow$ there are clouds.
There are clouds $\not\Rightarrow$ it is raining heavily.

12

Bases Other Than Ten

[EXPERIMENTS 97–100]

Experiment 97 : Numbers in Other Bases

One of the most important characteristics of our number system is the place value of digits *e.g.* in 555, the first digit from the left stands for 500, the second digit stands for 50 and the third digit stands for 5. The digit is the same, but its place values are 500, 50 and 5 according to the place it occupies.

The importance of place value, which was discovered by the ancient Hindus, from whom the Arab learnt it and passed it on to Europe, cannot be over-emphasized. Without this all our calculations would have been extremely complicated.

To understand the place value system, it is useful to study bases other than ten. The advantages are :

(*i*) The child understands the place value system better.

(*ii*) The child understands the operations of addition and multiplication better.

(*iii*) The child gets another opportunity to study the commutative, associative and distributive laws.

(*iv*) The child gets a number of interesting and (for brighter children) challenging problems to work with.

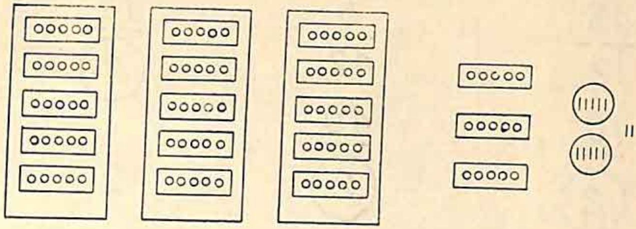
(*v*) The child gets a good deal of practice in problems of ordinary multiplication and division.

It is stated that man chose the base 10, because he had ten fingers on both hands. Now let us consider the type of numbers we would have, if man had decided to use fingers of one hand only. Instead of dealing with tens, he would be dealing in fives.

All that we have done in earlier experiments for base ten can now be done for base five.

Suppose a child is given a certain number, say 462, of sticks. Instead of bundles of ten, he makes now bundles of fives. He gets

92 bundles of fives and 2 single sticks. He again combines five bundles of fives to get bundles of twenty-fives. He gets 18 bundles each consisting of five bundles of fives. Two bundles of fives remain. He again combines each 5 of the bigger bundles into a still bigger bundle.



The child gets 3 bundles of one hundred twenty-five each, 3 bundles of twenty-five each, 2 bundles of five each and 2 bundles of 1 each. He writes, the number of sticks as $(3\ \ 3\ \ 22)_5$, the suffix denoting the base. Thus

$$(3322)_5 = 2+2\times5+3\times5^2+3\times5^3 = (462)_{10}$$

In general

$$(f\ e\ d\ c\ b\ a)_5 = (a+5\times b+25\times c+125\times d+625\times e+3125\times f)_{10}$$

It is obvious that each of a, b, c, d, ... is 0 or 1 or 2 or 3 or 4.

Similarly,

$$(f\ e\ d\ c\ b\ a)_2 = (a+2\times b+4\times c+8\times d+16\times e+32\times f)_{10}$$

Here each of the digits is 0 or 1.

In the same way

$$(d\ c\ b\ a)_7 = (a+7\times b+49\times c+343\times d)_{10}$$

Here each of the digits is 0 or 1 or 2 or 3 or 4 or 5 or 6. Again

$$(d\ c\ b\ a)_{12} = (a+12\times b+144\times c+1728\times d)_{10}.$$

Here each of the digits is 0 or 1 or 2 or 3 or 4 or 5 or 6 or 7 or 8 or 9 or t or e. The child can express a number in any base into a number to the base 10. This incidentally gives him practice in multiplication also.

Conversely to express any number in base ten into number to any given base, we proceed as in the experiment with sticks. Thus to express $(462)_{10}$ to base 5, we divide 462 by 5. The quotient is 92 and remainder is 2. 2 gives the first digit. Next divide 92 by 5. The quotient is 18 and the remainder is 2. This remainder gives the second digit from the right. Next divide 18 by 5 to get 3 as

quotient and 3 as remainder. We get the next two digits. The operation can be represented as follows :

$$
\begin{array}{l}
5\overline{)462}(92 \\
\quad \underline{45} \\
\quad 12 \\
\quad \underline{10} \\
\quad 2
\end{array}
\qquad
\begin{array}{l}
5\overline{)92}(18 \\
\quad \underline{5} \\
\quad 42 \\
\quad \underline{40} \\
\quad 2
\end{array}
\qquad
\begin{array}{l}
5\overline{)18}\ (3 \\
\quad \underline{15} \\
\quad 3
\end{array}
$$

The operations can be simplified as follows :

$$
\begin{array}{r|l}
5 & 462 \\ \hline
5 & 92 - 2 \\ \hline
5 & 18 - 3 \\ \hline
 & 3 - 3
\end{array}
$$

Similarly to express 4652 into base 12, we have

$$
\begin{array}{r|l}
12 & 4652 \\ \hline
12 & 387 - 8 \\ \hline
12 & 32 - 3 \\ \hline
 & 2 - 8
\end{array}
$$

$$(4652)_{10} = (2838)_{12}$$

This conversion gives the child sufficient practice in division.

To convert a number from one base to another base, it is desirable to convert from the first base to base ten and then from base ten to the second base. Thus suppose we want to convert $(356\,e)_{12}$ to base 5.

$$(356\,e)_{12} = (11 + 12 \times 6 + 144 \times 5 + 1728 \times 3)_{10}$$

$$= (5987)_{10}$$

5	5987	
5	1197	— 2
5	239	— 2
5	47	— 4
5	9	— 2
	1	— 4

$$= (142422)_5$$

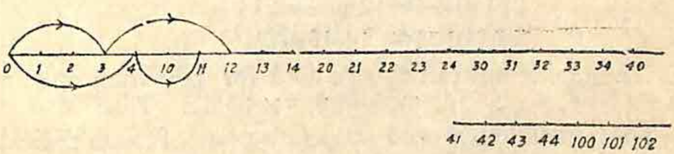
The children can do a large number of examples of such co-versions. Such problems give practice in multiplication and division which is more meaningful and interesting than a large drill in multiplication and division of arbitrary numbers.

Experiment 98 : Addition and Multiplication in Other Bases

For definiteness, let us consider base 5. The digits to be used are 0, 1, 2, 3 and 4. The numbers in the base will be

1	2	3	4	10
11	12	13	14	20
21	22	23	24	30
31	32	33	34	40
41	42	43	44	100
101	102	103	104	110
111	112	113	114	120

To form the addition table, we can use the number line.



$3 + 4 = 12$

$4 + 2 = 11$

In this, the child can complete the addition table for base 5.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

The table can also be used for subtraction. Thus

$$13-4=4,\ 10-1=4,\ 10-3=2,\ 11-2=4,\ 12-4=3.$$

To add two numbers we can use 'carrying' and for subtraction we can use 'borrowing'.

$$\begin{array}{rcccccc} & 1 & 1 & & 1 & 1 & \\ & 4 & 2 & 3 & 0 & 1 & 4 \\ + & 3 & 4 & 2 & 1 & 4 & 4 \\ \hline & 13 & 1\,2 & 1\,0 & 2 & 1\,1 & 1\,3 \end{array}$$

$$\begin{array}{rr} & 1\,3\,2\,0\,2\,1\,3 \\ - & 4\,2\,3\,0\,1\,4 \\ \hline & 3\,4\,2\,1\,4\,4 \\ \hline \end{array}$$

From the symmetry of the table about the diagonal, the commutativity of addition is easily verified. We can also verify associativity *e.g.*

$$(3+4)+4=12+\ 4=21$$
$$3+(4+4)=\ 3+13=21.$$

The number line can also be used for preparing multiplication table $4\times1=4$, $4\times2=13$, $4\times3=22$, $4\times4=31$. Thus

The cummutative and associative laws for multiplication and the distributive law for multiplication over addition can be easily verified. The table can also be used for division. Thus

$$31\div4=4,\ 14\div3=2$$

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

Multiplication and division can be done in the usual way by using the above tables.

$$\begin{array}{r} \text{3 3} \\ 2\,3\,4 \\ \times \quad 4 \\ \hline 2\,1\,0\,1 \end{array} \qquad \text{using } \begin{array}{r} 4\times4=31 \\ 4\times3=22 \\ 4\times2=13 \\ 22+3=30 \\ 13+3=21 \end{array}$$

$$\begin{array}{rl} 2\,3\,4 & \\ \hline 4)\;2\,1\,0\,1 & \\ 1\,3\,0\,0 & \quad 200\times4 \\ \hline 3\,0\,1 & \\ 2\,2\,0 & \quad 30\times4 \\ \hline 3\,1 & \\ 3\,1 & \quad 4\times4 \\ \hline 0 & \end{array}$$

Using $4\times2=13$,
$4\times3=22$, $\quad 4\times4=31$
$11-3=3$, $\quad 10-2=3$

The children need not do more complicated problems since the manipulative skill is required only in base 10.

The children should however prepare addition and multiplication tables in bases 2, 3, 4, 6, 7. Charts of addition and multiplication tables for bases 8, 9, 11, 12 can be displayed in the class and the various laws verified from them.

Experiment 99 : Binary Notation

The binary scale *i.e.*, use of base 2 is particularly important, since electronic computers use the arithmetic of numbers to base 2. The only digits used are 0 and 1 and the number line is

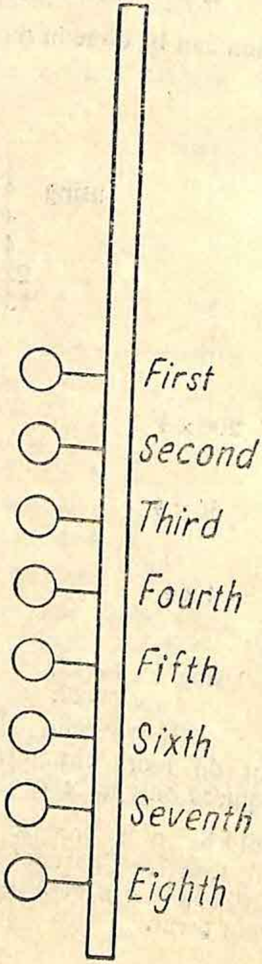
0 1 10 11 100 101 110 111 1000 1001 1010 1011 1100 1101 1110 1111

The addition and multiplication tables are :

+	0	1
0	0	1
1	1	10

x	0	1
0	0	0
1	0	1

To display numbers in the binary notation, we can use wooden boards with electric lamps fitted with different switches.



To show 110010, we light the second, fifth and sixth lamps. The children write the number and then find its representation in the scale of ten. Thus the above number is

$$0+2\times1+4\times0+8\times0+16\times1+32\times1=(44)_{10}$$

We can use this apparatus for coding messages *e.g.* the following correspondence can be established :

Number base 10	*Number in base 2*	*Letter represented*
1	1	A
2	10	B
3	11	C
4	100	D
5	101	E
6	110	F
7	111	G
8	1000	H
9	1001	I
10	1010	J
11	1011	K
12	1100	L
13	1101	M
14	1110	N
15	1111	O
16	10000	P
17	10001	Q
18	10010	R
19	10011	S
20	10100	T
21	10101	U
22	10110	V
23	10111	W
24	11000	X
25	11001	Y
26	11010	Z

The number 11111 may denote space between two words. Thus to give message MATHEMATICS IS FUN, we use a wooden board with five lamps. The first letter to be transmitted is M. Its binary number is 1101. The teacher lights first, third and fourth lamps, the children see the board and write 1101 in the first line and then M against it. When all the children have done this, the teacher switches off the lamps and lights the first bulb. The children write 1 and A against it. The teacher goes on transmitting M.A.T.H.E.M.A.T.I.C.S in succession. He then lights all bulbs. This shows the children

that the first word is over and the second is to begin. The teacher similarly transmits the other two words. In this experiment, the children will understand the binary scale as well as the reason for its usefulness in information theory and computers.

Alternatively the children can be given a code as follows :

10000, 101, 1, 11, 101

and asked to decipher it. They first read these in the scale of 10. They get 16, 5, 1, 3, 5. The sixteenth word of alphabet is P, the fifth word is E, first is A and the third is C so that the coded message is PEACE.

This message could also be coded on a peace of paper as follows by punching holes in the places given. The child writes 1 in place of every hole and 0 for other places on its right and interprets the message.

0				
		0		0
				0
			0	0
		0		0

10000	16	P
101	5	E
1	1	A
11	3	C
101	5	E

Experiment 100 : Other Illustrations of Other Bases

(*a*) The children may be asked to prepare a calendar of some month in some base say 5. The calendar of May 1969 wood look as follows :

MAY $(30334)_5$

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
—	—	—	—	1	2	3
4	10	11	12	13	14	20
21	22	23	24	30	31	32
33	34	40	41	42	43	44
100	101	102	103	104	110	111

(*b*) Two rulers to base 5 may be constructed.

0 1 2 3 4 10 11 12 13 14 20 21 22 23 24 30 31 32 33 34 40

To add and subtract numbers, the children may use the method of experiment 2.

(*c*) The children may try to develop tests of divisibility in various bases *e.g.*, in base 7.

$$49c+7b+a=(48c+6b)+(c+b+a)$$

$\therefore$ $c\ b\ a$ is divisible by 2 or 3 or 6 if $a+b+c$ is divisible by 2 or 3 or 6 respectively *e.g.* 456 is not divisible by 2 or 6 but is divisible by 3. This shows that while $(456)_{10}$ is divisible by 2, $(456)_7$ is not divisible by 2. The tests are different for different bases. The numbers which are 'even' in one base may look odd in another base.

(*d*) The children may find 'prime' numbers in other bases by using the method of the sieve of Eratosthenes.

(*e*) The children may find 'truth base' of statements *e.g.*

Statement	*Truth base*
I have 10 fingers in one hand	5
4+4=10	8
A quadrilateral has 10 sides	4
There are 10 months in a year	12
I study in class 10	Class of the student
A car has 11 wheels	3
2+2=5	No base.

Exercises

1. Make a list of all the symbols used in this book and explain each one of them.

2. Find 4+5, 5+4, 5+7, 7+5, (3+4)+5, 3+(4+5), (4+5)+7, 4+(5+7) by using each of the six methods of experiment 2. What patterns do you observe ?

3. Find 7−3, 10−8, 15−5, 16−7 by using each of the six methods of experiment 3.

4. Explain the importance of the place value system for mathematics, science and technology.

5. Prepare the number board made of plywood or cardboard with numbers 1 to 100 arranged in ten rows and ten columns. Discuss all the patterns you observe in its rows, columns and diagonals. Where do multiples of 2, 3, 4, 5, 6, 7 and 9 occur on this board ?

6. Can you suggest some other advantages (or disadvantages) of the system of naming numbers in experiment 6. Will the new names for numbers require less or more letters in printing ? Read the following passage (using new names) :

   11, 13, 17, 19 are prime numbers, 12, 14, 16, 18 are even numbers, 15 is neither a prime number nor an even number.

7. (*a*) Write old names and new names for numbers 1—100 in Hindi in parallel columns.

   (*b*) Explain the pronunciation of the old names for numbers in Hindi.

   (*c*) Write names for numbers 1 to 100 in all the languages of India and find which of these languages already use the principle stated in experiments 6 and 7.

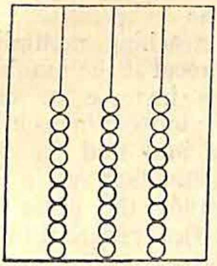
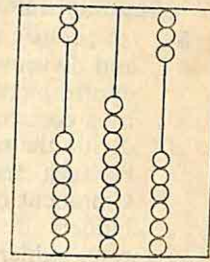
8. In Egyptian symbolism, pictures of ropes, flowers, lives etc. represented various numbers. The most commonly used symbols were the following :

1	10	100	1000	10000	100000

Stroke	Archor Heel bone	Coiled rope	Lotus flower	Pointed finger	Tadpole

Using these symbols, represent the following numbers :
(*i*) 155, (*ii*) 3456, (*iii*) 203123 (*iv*) 99999.

9. Make a simple sketch to show the beads arrangement on a nine-bead abacus of (*i*) 354 (*ii*) 456 (*iii*) 2345

If  is the abacus in the zero state and  represents 203

10. Translate the following Roman numbers into Hindi-Arabic numbers, remembering that V, X, L, C, D, M stand for 5, 10, 50, 100, 500 and 1000 respectively and a bar over a letter indicates its value multiplied by 1000 :

(*i*) XXVII (*ii*) XLVI (*iii*) CXCV
(*iv*) $\overline{C}$ (*v*) $\overline{D}$ (*vi*) $\overline{M}$
(*vii*) $\overline{DC}$V (*viii*) $\overline{MDC}$LV.

11. Represent in Roman notation the numbers represented in Hindi-Arabic notation below :
(*i*) 356 (*ii*) 4036 (*iii*) 400526 (*iv*) 30006784.

12. Solve the following problems without using Hindi-Arabic numbers :
(*i*) MDV − CCLX (*ii*) MCDV + CCXXIV.
(*iii*) XLIX + VII (*iv*) CLV × XXVII.

13. Justify the answer given after every question :

(*i*)	Is 4 larger than 5 ?	No
(*ii*)	Is '9' smaller than '8' ?	Yes
(*iii*)	Does 'four' have four letters ?	Yes
(*iv*)	Does five have four letters ?	No
(*v*)	Does '53' consist of 2 numbers ?	No
(*vi*)	Does '2' plus '1' equal 5 ?	No
(*vii*)	Does 2 plus 3 equal 5 ?	Yes
(*viii*)	Can you drink 'water' ?	No
(*ix*)	Can you write 'red' with black lead pencil ?	Yes
(*x*)	Do we write 2 first in writing '26' ?	No

14. Write all numbers from 1 to 50 using four fours only and the symbols +, −, ×, ÷, 5 and the symbols for decimal and recurring decimal *e.g.* $1 = 44 \div 44$, $2 = 4 \div 4 + 4 \div 4$, $3 = \sqrt{4} + \sqrt{4} - 4 \div 4$.

15. Explain the use of the concept of 'remaming' in the context of multiplication and division problems.

16. (*a*) Modify the game of five ladders for teaching multiplication and division. (*b*) In each ladder of experiment 16 the numbers are in arithmetic progression *i.e.* the numbers increase or decrease by a constant amount. Find the sum of numbers in each ladder, divide the sum by the number of terms and find the relation between the quotient and the sum of the first and last terms. Comment on the result obtained. (*c*) Study the game of five ladders if the numbers are in geometrical progression *i.e.* the each ladder, they increase or decrease by a constant ratio.

17. Verify that the following are magic squares

(*i*)

$n+7$	n	$n+5$
$n+2$	$n+4$	$n+6$
$n+3$	$n+8$	$n+1$

(*ii*)

16	2	3	13
5	11	10	8
9	7	6	12
9	14	15	1

(*iii*)

67	1	43
13	37	61
31	73	7

(*iv*)

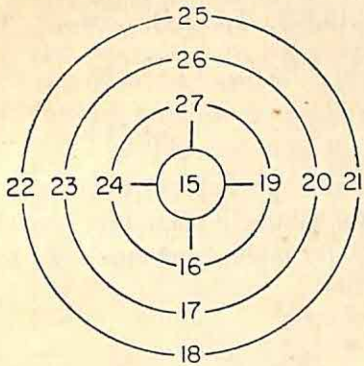
3	71	5	23
53	11	57	1
17	13	41	31
29	7	19	47

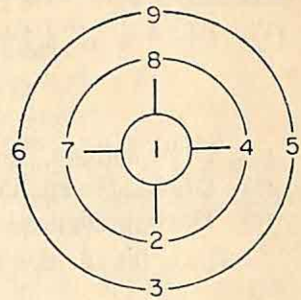
(*v*)

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Here the squares (*iii*) and (*iv*) contain only prime numbers and 1 while the square (*v*) is a famous square occuring in a painting by Durer in 1514 and has many interesting properties (*c.f.* experiment)

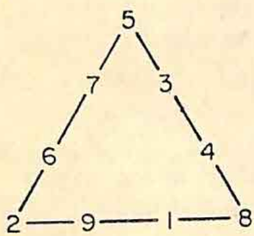
18. Verify that the following are magic circles *i.e.* the sum of numbers on each circle and on each diameter is the same.

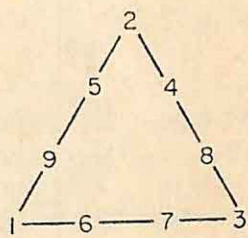


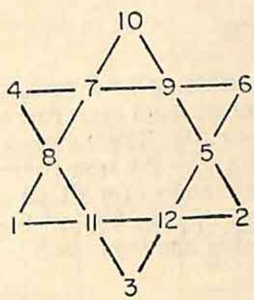


Construct more magic circles.

19. Verify that the following are magic figures :







Get more magic figures out of these.

20. Write the three numbers in each of the following sequences :

(*i*) 30, 27, 24, 21, 18, ...

(*ii*) 5, 9, 13, 17, 21, ...

(*iii*) 1, 2, 4, 8, 16, ...

(*iv*) 1.2, 2.3, 3.4, 4.5, 5.6, ...

(v) $\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\ldots\ldots\ldots$

(vi) 1.1, 3 2. 5.4, 7.8, 9.16, ...

(vii) $\frac{1}{1}, \frac{2}{2}, \frac{4}{3}, \frac{8}{4}, \frac{16}{5}, \ldots\ldots\ldots$

(viii) (,2,3), (2,3,4), (3,4,5), (4,5,6), (5,6,7), ...

(ix) (1,2,4) (2,4,8) (4,8,16), (8,16,32) (16,32,64), ...

(x) $1+2+\frac{1}{3},\ 2+4+\frac{1}{6},\ 3+8+\frac{1}{11},\ 4+16+\frac{1}{20},$

$5+32+\frac{1}{37}, \ldots\ldots$

21. (a) Draw addition clocks for adding numbers 1 to 50.

(b) Draw subtraction clock for subtracting numbers 1 to 50.

(c) Draw multiplication clock for multiplying numbers 1 to 10.

22. Complete the addition table

+	92	143	194	245	296
100					396
150				395	
200			394		
250		393			
300	392				

Verify that in the resulting 5×5 matrix, if we take 5 elements, such that one comes from each row and one comes from each column, the sum is 1970. There can be $5^2 \times 4^2 \times 3^2 \times 2^2 \times 1^2$ such sums (why). Explain why the sum comes out to be 1970 in every case. From other matrices for which the sum is 1969, 1947, 1950 and the year of your birth. Find matrices for which the product of 5 elements one from each row and each column, is 14400.

23. The above problem can be used for a modified tic-tac-toe game to be played by the children. Two children play alternately, only the numbers on the top and left are written first. Each child writes one of the twenty-five sums and draws a circle or a square round it. If he makes a mistake and the other child finds it out, he loses his turn. The child who gets five numbers correct in a straight line first wins the game. Similar games can be played for multiplication and subtraction.

24. In a addition problem □ + △ = ▱
   (*i*) What happens to ▱ if both □ and △ are doubled ?
   (*ii*) What happens to □ if both ▱ and △ are doubled ?
   (*iii*) What is ▱ − △ equal to ?
   (*iv*) What happens if ▽ is subtracted from both □ and △ ?

25. In a subtraction problem, □ − △ = ▱,
   (*i*) What happens to ▱ if both □ and △ are doubled ?
   (*ii*) What happens to ▱ if both □ and △ are increased by △ ?
   (*iii*) What is △ + ▱ equal to ?
   (*iv*) What happens to ▱ if □ is trebled ?

26. In the multiplication problem □ × △ = ▱,
   (*i*) What happens to ▱ if □ is doubled ?
   (*ii*) What happens to ▱ if both □ and △ are doubled ?
   (*iii*) What happens to ▱ if □ is doubled and △ is halved ?
   (*iv*) What happens to ▱ if □ is doubled and △ is trebled ?
   (*v*) What happens to ▱ if □ is increased by 3 ?
   (*vi*) What is △ when □ = ▱ ?

27. For all two-digit numbers from 11 to 99, find the number of steps required to reach a symmetrical number by using the reversed addition process.

28. For the reversed subtraction process, find all the cycles for two digit, three digit and four digit numbers.

29. Will the multiplication of a number by 11, 101, 1001 etc. always lead to a symmetrical number ?

30. Let $\bar{1}=9$, $\bar{2}=8$, $\bar{3}=7$, $\bar{4}=6$ so that $\bar{1}\ \bar{2}\ \bar{3}\ \bar{4}\ 5\ 4$ is the number 9 8 7 6 5 4. The digits used in this process will be 0, 1, 2, 3, 4, 5, $\bar{4}$, $\bar{3}$, $\bar{2}$, $\bar{1}$. Investigate addition, subtraction and multiplication problems in this notation.

31. Multiply 7 by 8 by using each of the methods of experiments 31—36.

32. Divide 42 by 7 by using each of the methods of experiments 37—41.

33. Given the addition and multiplication tables below :

+	0	1	∧	△	□
0	0	1	∧	△	□
1	1	∧	△	□	10
∧	∧	△	□	10	11
△	∧	□	10	11	1∧
□	□	10	11	1∧	1△

X	0	1	∧	△	□
0	0	0	0	0	0
1	0	1	∧	△	□
∧	0	∧	□	11	1△
△	0	△	11	1□	∧∧
□	0	□	1△	∧∧	△1

(a) Solve the following problems :

(i) $$\begin{array}{r} \triangle\triangle \\ +\ \square\wedge \\ \hline \end{array}$$ (ii) $$\begin{array}{r} \square\square\wedge \\ -\ \wedge\wedge\triangle \\ \hline \end{array}$$

(iii) $$\begin{array}{r} \triangle\square\wedge \\ \times\ \triangle \\ \hline \end{array}$$ (iv) $$\frac{\wedge\ 1}{\square}$$

(b) Interpret $\wedge$ as 2, $\triangle$ as 3 and $\square$ as 4 and rewrite the above tables. What do you find (*c.f.* experiment 97)? Do you see any justification for the notation used ?

(c) Rewrite the tables of experiment 93 in this notation.

34. Prepare the multiplication tables of 2 to 10 by using each of the methods of experiments 31—36.

35. Write all the division facts in the form

$$\frac{\square}{\triangle}=\varnothing \quad \text{or} \quad \frac{\square}{\triangle}>\varnothing \quad \text{or} \quad \frac{\square}{\triangle}<\varnothing$$

where $\square$ is any number from 1 to 100 and $\triangle$ is a number smaller than $\square$ and all numbers considered are natural numbers. How many facts of the first type do you get ?

36. Find the number of solutions in natural numbers of the following equations :

(i) $x+y=n$ $(n=2, 3, 4, 5, 6, \ldots\ldots\ldots\ldots)$

(ii) $x+y+z=n$ $(n=3, 4, 5, 6, 7, \ldots\ldots\ldots\ldots)$

(iii) $x+y+z+u=n$ $(n=4, 5, 6, 7, 8, \ldots\ldots\ldots\ldots)$

Verify that the number of solution is

$$\frac{n-1}{1} \quad \frac{(n-1)(n-2)}{1,\ 2} \quad \frac{(n-1)(n-2)(n-3)}{1,\ 2,\ 3}.$$

Generalise your results.

37. Find the number of solutions in natural numbers of the following inequality

$$x+y\leqslant n \qquad (n=2, 3, 4, 5, 6, \ldots\ldots\ldots\ldots)$$

Generalise your result.

38. Find the number of solutions in non-negative integers of the equations of exercise 36 and comment on the pattern you get.

39. Write numerals for letters so that the following become correct problems :

$$\begin{array}{r} \text{S E N D} \\ +\ \text{M O R E} \\ \hline \text{M O N E Y} \end{array} \qquad \begin{array}{r} \text{S E N D} \\ \text{M O R E} \\ +\ \text{G O L D} \\ \hline \text{M O N E Y} \end{array}$$

40. Write numerals for letters so that the following become correct problems :

$$\begin{array}{r} \text{F U R} \\ \text{D O G} \\ \hline \text{A G E R} \\ \text{D R I P} \\ \text{F U R} \\ \hline \text{R I P E R} \end{array} \qquad \begin{array}{r} \text{N I N E} \\ -\ \text{T E N} \\ \hline \text{T W O} \end{array} \qquad \begin{array}{r} \text{S E V E N} \\ -\ \text{N I N E} \\ \hline \text{T W O} \end{array}$$

41. Select a number for 1, 2, 3, 4, 5, 6, 7, 9 and multiply it by 9. Now multiply 12345679 by the product. Comment on the pattern you get.

42. (*a*) Show that the remainder obtained on dividing a number by 9 is the same as the remainder obtained by dividing the sum of its digits by 9.

(*b*) Show that the remainder obtained on dividing the sum or difference or product of two numbers by 9 is the same as the remainder obtained by dividing respectively sum or difference or product of the remainders by 9. Find the remainders on dividing 987654 and 345671 by 9. Find the sum or difference or product of these numbers and the remainders obtained hy dividing these numbers by 9.

Verify (*a*) and (*b*)

43. Which of the following sets are closed ?

(*i*) the set of even numbers under addition.
(*ii*) the set of odd numbers under addition.
(*iii*) the set of even numbers under multiplication.
(*iv*) the set of odd numbers undeı multiplication.
(*v*) the set of multiples of 3 under addition and multiplication.

44. In which of the following examples is the commutative property valid ?

(*i*) Taking a shower, removing clothes.
(*ii*) Putting on shoes, putting on socks.
(*iii*) Brushing your teeth, combing your hair.
(*iv*) Messaging hair oil, combing the hair.
(*v*) Adding sulphuric acid to water, adding water to sulphuric acid.
(*vi*) Scraping the door, painting the door.

Give more examples of the same type.

45. Use lattice multiplication method to find

$23 \times 45, \quad 456 \times 678, \quad 2345 \times 4567.$

46. State the rule which would enable us to decide which of the two given numbers *abcdef* and *pqrstu* is greater where letters denote digits. Does the same rule hold in scales other than ten ?

47. Rewrite experiments 47 – 50 by using the concept of 'renaming'.

48. Discuss multiplication of two-place decimals by natural numbers by using the concept of rupees and paise or of metres and centimetres and the multiplication of three-place decimals by natural numbers by using the concept of kilometres and metres.

49. Discuss addition and subtraction of five-place decimals and their multiplication by natural numbers by using the concept of kilometres, metres and centimetres.

50. Give a rule for deciding as to which of the numbers *abc.def*, *pqr.stu* is greater ?

51. Does the commutative law hold for the following operations in N ?

(*i*) $a \ 0 \ b = b$
(*ii*) $a \ 0 \ b = a + b + ab$
(*iii*) $a \ 0 \ b = a^2 + b^2$
(*iv*) $a \ 0 \ b = a + b$.

52. For which of the above four operations, does the associative law hold ?

53. In N, give examples of operations which are
(*i*) Commutative but not associative.
(*ii*) Associative but not commutative.
(*iii*) Both commutative and associative.
(*iv*) Neither commutative nor associative.

54. Give all possible values of $2 \bigstar 3 \bigstar 4 \bigstar 3 \bigstar 2$ if $a \bigstar b = a^b$.

55. In a set an operation 0 is said to distribute another operation $\bigstar$ on the right if

$$a \ 0 \ (b \bigstar c) = (a \ 0 \ b) \bigstar (a \ 0 \ c)$$

and 0 is said to distribute $\bigstar$ on the left if

$$(b \bigstar c) \ 0 \ a = (b \ 0 \ a) \bigstar (c \ 0 \ a)$$

Which of the operations $+, -, \times, \div$ distribute the other in N ?

56. How many distributive laws are possible with 5 or (6) operations ? which of these are true for the operations $+, -, \times, \div$, $0, \bigstar$ in N where $a \ 0 \ b = a^b$ and $a \bigstar b = a + b + ab$?

57. Find 546×653 and explain which laws you use in this process.

58. Divide 567825 by 15 and explain which laws you use in this process.

59. Use the process of successive approximation to find the result of dividing 31215 by 15.

60. Fill in the proper operation signs :

5		6		2	=	8
3		3		6	=	3
5		6		8	=	19
=		=		=		=
3		3		4	=	5

61. Find simple number names for

(*i*) $^{+}3\times{}^{-}4+{}^{-}6\div{}^{+}2-{}^{+}1$

(*ii*) $\overleftarrow{4}\times\overrightarrow{3}+\overrightarrow{6}\div\overleftarrow{2}-\overrightarrow{1}$

(*iii*) $3\times(-4)+(-6)\div 2-1$

Write (*i*) in terms of mirror numbers.

62. Show that in N every number has successor, but does not necessarily have a predecessor, but in I every integer has both a successor and a predecessor.

63. Throw a coin 20 times and move on the number line. Repeat this experiment 100 times and find the frequencies with which the counter is at $-20, -19, \ldots\ldots 0, 1, \ldots\ldots, 20$.

64. Verify the commutative, associative and distributive laws in the system of integers by at least five examples of each law.

65. In I, show that

(*i*) If $a>b$ and $b>c$, then $a>c$.

(*ii*) If $a<b$ and $b<c$, then $a<c$.

(*iii*) If $a\geqslant b$ and $b\geqslant c$, then $a\geqslant c$.

(*iv*) If $a\leqslant b$ and $b\leqslant c$, then $a\leqslant c$.

(*v*) If a and b are any two numbers, either

$$a>b \text{ or } a=b \text{ or } b>a.$$

66. Given numbers 1, 4, 8, 12 and symbols $+$, $-$, $\div$, $\times$, $>$, $=$; how many true sentences can you make ?

67. If A is father of B, is B father of A ?

If A is brother of B, is B brother of A ?

If A is cousin of B, is B cousin of A ?

If A likes B, B likes C, does A necessarily like C ?

If A=B, B=C, is A necessarily equal to C ?

If A=B (mod 7) B=C (mod 7) is A=C (mod 7) ?

68. Extend the table of experiment 68 for multipliers and multiplicand going from -10 to 10.

69. Illustrate the laws of the system of integers geometrically with the help of the number line, taking at least two different examples for each case.

70. A drunkard moves on the number line between -5 and 5. There are walls at -5 and 5 when he reaches 5, he is sent back to 4 and when he reaches -5, he is sent back to -4.

Find where the drunkard is after 10 steps. Repeat the experiment 100 times and find the relative frequencies of the positions

$$-4, -3, -2, -1, 0, 1, 2, 3, 4.$$

71. Verify for particular sets :
   (i) $A \cup B = B \cup A$
   (ii) $A \cap B = B \cap A$
   (iii) $(A \cup B) \cup C = A \cup (B \cup C)$
   (iv) $(A \cap B) \cap C = A \cap (B \cap C)$
   (v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (vii) $A \cup \phi = A$
   (viii) $A \cap \phi = \phi$.

72. Show that the following are null sets :
   (i) The set of all natural numbers which are not equal to themselves.
   (ii) The set of integers whose squares are negative.
   (iii) The set of all hundred-storeyed buildings in India.

   Give more examples of null sets.

73. Give a set of numbers which is
   (i) Closed for addition and multiplication, but is not closed for subtraction and division.
   (ii) Closed for multiplication but not for the other three operations.
   (iii) Closed for subtraction, but not for the other three operations.

74. Distinguish between a line, a line segment and a ray.

75. Discuss the possible intersection sets of
   (i) two lines (ii) three lines (iii) two planes
   (iv) three planes (v) three planes and a line
   (vi) three planes and two lines (vii) four planes and a line.

76. Draw curves and polygons which are
   (i) open (ii) closed
   (iii) closed but not simple (iv) simple closed
   (v) simple closed but not convex
   (vi) simple closed and convex.

   Find the shortest path between any two points of (a) a convex region (b) a non-convex region.

77. Make cardboard and plasticine models of about twenty different polyhedra and verify Euler's formula in each case.

78. Find the H.C.F. and L.C.M. of 14, 56, 84 from first principles.

79. Find the truth sets of the following open sentences when the replacement set is N
   (i) $\square \leqslant 6$
   (ii) $\square + \triangle = 4$, $\square + \triangle = 5$, $\square + \triangle = 6$
   (iii) $\square + \triangle < 6$, $\square + \square \leqslant 6$

(*iv*) $\square - \triangle < 4$, $\square - \triangle \geqslant 4$

(*v*) $\square \times \triangle = 24$

(*vi*) $\square \times \triangle \leqslant 36$.

80. Find the truth sets of the above sentences when the replacement set is I and draw their graphs.

81. Connect the consecutive points in the following set by line segments :

(2, 3), (7, 3), (10, 1), (8, 4), (10, 6), (8, 6), (10, 10), (7, 7), (5, 10), (5, 6), (3, 6), (3, 9), (1, 9), (1, 8), (2, 8), (2, 3).

Draw also the line segments joining (3, 1) to (4, 3) and (5, 3) to (6, 1). What figure do you get ?

82. If the replacement set is I, draw the graphs of truth sets of the following open sentences :

(*i*) $|x| = 4$ (*ii*) $y = |x|$

(*iii*) $|x| + |y| = 5$ (*iv*) $y > |x|$

where $|x|$ denotes the absolute value of x so that

$$|3| = 3, \quad |-3| = 3.$$

83. In I, find the truth sets of the following open sentences and draw their graphs :

(*i*) $x^2+y^2=25$ (*ii*) $x^2+y^2=100$.

On what curves do the points lie ?

84. Solve the equations graphically :

(*i*) $x+y=13$, $xy=36$

(*ii*) $x+y=13$, $x-y=1$

(*iii*) $x+y=7$, $x^2+y^2=25$.

85. Take a graph paper. Draw a number of polygons with vertices at the lattice points. For each polygon, find

(*i*) The number of lattice points on the polygon.

(*ii*) The number of lattice points inside the polygon.

(*iii*) The number of squares which are completely within the polygon or more than half inside it.

Do you find any relation between sum of (*i*) and (*ii*) and (*iii*) ?

86. Draw a number of polygons in the first quadrant and with vertices at lattice points. Find the values of $x+y$, $x-y$, $2x+3y$ at all the lattice points within and on each polygon. Where is the value maximum in each case ?

87. Find the truth set of

$$\square + \triangle + \varodot = 6$$

and try to represent it graphically.

88. Find the truth sets of

(*i*) $\square + \triangle + \varodot = 6$, $\square - \triangle + \varodot = 2$, $-\square + \triangle + \varodot = 4$.

(*ii*) $\square + \triangle + \varodot = 6$, $\square - \triangle + \varodot = 2$, $\square + \varodot = 4$.

89. (*a*) Find all prime numbers less than 150.

(*b*) Find all prime factors of 24, 36, 48, 56. Are the factors unique?

90. Is n^2-n+41 a prime number for all values of n ?

91. Find the next three numbers in the following sequences :

(*i*) 1, 3, 4, 7, 11, 18, 29, 47, 76,......

(*ii*) 1, 3, 8, 22, 60, 164, 448,......

(*iii*) 1, 1, 1, 3, 5, 9, 17, 31, 57, 105,......

92. Find the remainders when the following numbers are divided by 5, 7, 9, 11, 13, and 17 :

643251, 32571, 45678.

93. Write addition and multiplication tables for arithmetic modulo 5 and verify all the properties mentioned in experiment 93.

94. Write addition and multiplication tables for arithmetic modulo 6 and modulo 8. Which properties here differ for those of experiment 93 ? Do N and I satisfy all these properties ?

95. Given $1+2+3+4+5+6=21$, deduce as many results from it as you can.

96. Give ten examples of the following types :

(*a*) $P \Rightarrow Q$ and $Q \Rightarrow P$

(*b*) $P \Rightarrow Q$ and $Q \Rightarrow P$

(*c*) $P \Rightarrow Q$ and $Q \Rightarrow P$

97. Fill in the blanks :

$(3456)_{10}=(\quad)_5$

$(5678)_{10}=(\quad)_{12}$

$(2341)_5=(\quad)_8$.

98. Find

(*i*) $(1234)_5 \times (2341)_5$

(*ii*) $(23456)_7+(34562)_7-(12345)_7$

(*iii*) $(1100100)_2+(101)_2$

99. Find binary codes for the following :

(*i*) MATHEMATICS IS INTERESTING.

(*ii*) MATHEMATICS IS A DYNAMICAL INTELLI CTUAL ENTERPRISE.

(*iii*) ABSTRACTION PRECISION ELEGANCE AND DEPTH ARE CHARACTERISTICS OF MATHEMATICS.

100. Prepare calendars for 1970 in bases 5, 7 and 12.

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